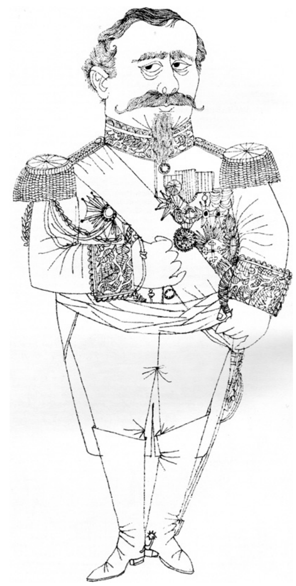
The Bourbaki World

27 April 2020

Jim Stevenson

I thought it would be interesting to present a recent entry in the mathematician John Baez's Diary ([1]) on some extremes in mathematics from the Bourbaki school, namely, how many symbols it would take to define the number "1."



Scientific American

I don't know if the "mathematician" Nicolas Bourbaki holds any significance for students today, but in my time (math graduate school in the 1960s) the Bourbaki approach seemed to permeate everything.

My first exposure to Bourbaki was as a humorous figure described by Paul Halmos in his 1957 article in the *Scientific American* ([2])—the humor being that Bourbaki did not exist. (More details can be found by Googling or at *Wikipedia* ([3]).) As Halmos wrote:

> One of the legends surrounding the name is that about 25 or 30 years ago first-year students at the Ecole Normale Superieure (where most French mathematicians get their training) were annually exposed to a lecture by a distinguished visitor named Nicolas Bourbaki, who was in fact an amateur actor disguised in a patriarchal beard, and whose lecture was a masterful piece of mathematical double-talk. It is necessary to insert a word of warning about the unreliability of most Bourbaki stories. While the members of this cryptic organization have taken no blood oath of secrecy, most of them are so amused by their own joke that their stories about themselves are intentionally conflicting and apocryphal. ([2])

Nicholas Bourbaki was the pseudonym for a group of French mathematicians who wished to write a treatise which would be, as Halmos stated, "a survey of all mathematics from a sophisticated point of view" ([2]). Further:

> The main features of the Bourbaki approach are a radical attitude about the right order for doing things, a dogmatic insistence on a privately invented terminology, a clean and economical organization of ideas and a style of presentation which is so bent on saying everything that it leaves nothing to the imagination … ([2])

A bona fide historian of mathematics could probably describe the situation more accurately, but from my personal experience I felt the Bourbaki approach heavily influenced the abstract definition-theorem-proof-remark style of presentation that our graduate classes assumed. Rather than arrive at new mathematical ideas inductively through examples and difficulties that needed to be resolved, we were given clean abstract definitions, followed by chains of theorems and proofs that seemed to float disconnected from anything we knew. The examples came later. Mathematics was not presented as it might be discovered but rather as a tidy flow of Euclidean logic. I believe this approach is dubbed the "axiomatic method."

Refreshing my memory of Bourbaki has led me to an essay cited by Halmos, "The Architecture of Mathematics,"([4] (1950)) where Bourbaki explained their approach, including a corroboration of my impression of the axiomatic method. Cast in the best light, it is an exiting view, one for which in fact I have provided some examples: "The Essence of Mathematics,"[1] where seemingly disparate

[1] http://josmfs.net/2019/03/03/the-essence-of-mathematics/

games yield tic-tac-toe as a unifying abstraction, and "Point Set Toplogy,"[2] where questions of infinite processes, limits, and nearness lead from numbers to points in space, to collections of functions, to metric spaces, and then to topologies as a final unifying abstraction. This collecting together of seemingly unrelated entities into a unifying abstraction was a defining characteristic of 20th century mathematics and very exciting. But to my mind, the axiomatic method took a short-cut—it skipped the presentation of the disparate entities and went directly to the abstraction and its study. Later on it introduced the scattered instances of the abstraction as "examples". But I contend the human mind *learns* though examples by an inductive process. The result is a collection of concrete images that motivates the exploration of the abstraction and its consequences. Without those preliminaries, studying abstractions for their own sake is a difficult task. Many is the time Feynman got pleasure in tweaking some physicist with a concrete situation that the physicist did not realize was a special case of some abstraction. Somehow studying only abstractions one looses touch with the concrete.

But I digress. I now present the logical extreme which is Bourbaki—John Baez's Diary:

April 14, 2020

The French mathematicians who went under the pseudonym Nicolas Bourbaki did a lot of good things—but not so much in the foundations of mathematics. Adrian Mathias, a student of John Conway, showed their definition of "1" would be incredibly long, written out in full.[3]

A term of length 4,523,659,424,929

A. R. D. Mathias

Universidad de los Andes, Santa Fé de Bogotá and Humboldt Universität zu Berlin

Abstract Bourbaki suggest that their definition of the number 1 runs to some tens of thousands of symbols. We show that that is a considerable under-estimate, the true number of symbols being that in the title, not counting 1,179,618,517,981 disambiguatory links.

One reason is that their definition of the number 1 is complicated in the first place. Here it is. I don't understand it. Do you?[4]

Bourbaki's abbreviated definition of 1

Chapters I and II of Bourbaki's *Théorie des Ensembles* were published in 1954, and Chapter III in 1956. Among the primitive signs used was a reverse C, standing presumably for "couple", to denote the ordered pair of two objects. Being typographically unable to reproduce that symbol, we use instead the symbol •. With that change, the footnote on page 55 of Chapter III reads

Bien entendu, il ne faut pas confondre le terme mathématique désigné (chap. I, §1, n° 1) par le symbole "1" et le mot "un" du langage ordinaire. Le terme désigneé par "1" est égal, en vertu de la définition donnée ci-dessus, au terme désigné par le symbole

$$\tau_Z((\exists u)(\exists U)(u = (U, \{\varnothing\}, Z) \text{ et } U \subset \{\varnothing\} \times Z \text{ et } (\forall x)((x \in \{\varnothing\}) \implies (\exists y)((x, y) \in U))$$
$$\text{et } (\forall x)(\forall y)(\forall y')(((x, y) \in U \text{ et } (x, y') \in U) \implies (y = y')) \text{ et } (\forall y)((y \in Z) \implies (\exists x)((x, y) \in U)))).$$

Une estimation grossière montre que le terme ainsi désigné est un assemblage de plusieurs dizaines de milliers de signes (chacun de ces signes étant l'un des signes τ, $\square$, $\vee$, $\neg$, $=$, $\in$, •).

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2 http://josmfs.net/2018/12/28/point-set-topology/
3 https://www.dpmms.cam.ac.uk/~ardm/inefff.pdf
4 https://www.dpmms.cam.ac.uk/~ardm/inefff.pdf

But worse, they don't take ∃, "there exists", as primitive. Instead they *define* it—in a truly wretched way. They use a version of Hilbert's "choice operator". For any formula $\Phi(x)$ they define a quantity that's a choice of x making $\Phi(x)$ true if such an choice exists, and just anything otherwise. Then they define $\exists x\Phi(x)$ to mean Φ holds for this choice.[5]

> Quantifiers are introduced as follows:
>
> B·5 DEFINITION $(\exists x)R$ is $(\tau_x(R) \mid x)R$;
>
> B·6 DEFINITION $(\forall x)R$ is $\neg(\exists x)\neg R$.
>
> Thus in this formalism quantifiers are not primitive. Informally, the idea is to choose at the outset, for any formula $\Phi(x)$ a witness, some a such that $\Phi(a)$; call it $\tau_x\Phi$. If there is no such witness, let $\tau_x\Phi$ be anything you like, say the empty set.

This builds the axiom of choice into the definition of ∃ and ∀. Worse, their implementation of this idea leads to huge formulas. And in the 1970 edition, things got much worse! [6]

> In the combined 1970 edition of chapters I to IV, Bourbaki revert to the definition familiar to set theorists of the ordered pair of x and y as $\{\{x\}, \{x, y\}\}$. The corresponding footnote, on page E III 24 of that edition, is almost identical to the original, the only differences being the omission of a primitive symbol (the reverse C) for ordered pair, and the reference to Chapter I appearing more simply as (I, p.15).
>
> Though there are good reasons for that change, it would mean, given the commitment of Bourbaki to the τ operator, an enormous increase in the number of symbols in their definition of the term 1, for $\bullet xy$, instead of being of length 3 with one occurrence each of x and y, and no link, will be of length 4,545, with 336 occurrences of x, 196 occurrences of y and 1,114 links. $X \times Y$ will now be of length roughly $3{\cdot}1845912 \times 10^{18}$, with $1{\cdot}15067 \times 10^{18}$ links, and $6{\cdot}982221 \times 10^{14}$ occurrences each of X and of Y, and a program in Allegro Common Lisp written by Solovay yields these exact figures:
>
> 7·0 PROPOSITION *If the ordered pair (x, y) is introduced by definition rather than taken as a primitive, the term defining 1 will have 2409875496393137472149767527877436912979508338752092897 symbols, with 871880233733949069946182804910912227472430953034182177 links.*
>
> At 80 symbols per line, 50 lines per page, 1,000 pages per book, the shorter version would occupy more than a million books, and the longer, 6×10^{47} books.

You can read Mathias' paper here:

Adrian R. D. Mathias, "A term of length 4,523,659,424,929"[7] *Synthese* **133** (2002), 75–86.

For my own overview, see:

John Baez, "Bigness (Part 1)",[8] *Azimuth*, April 13, 2020.

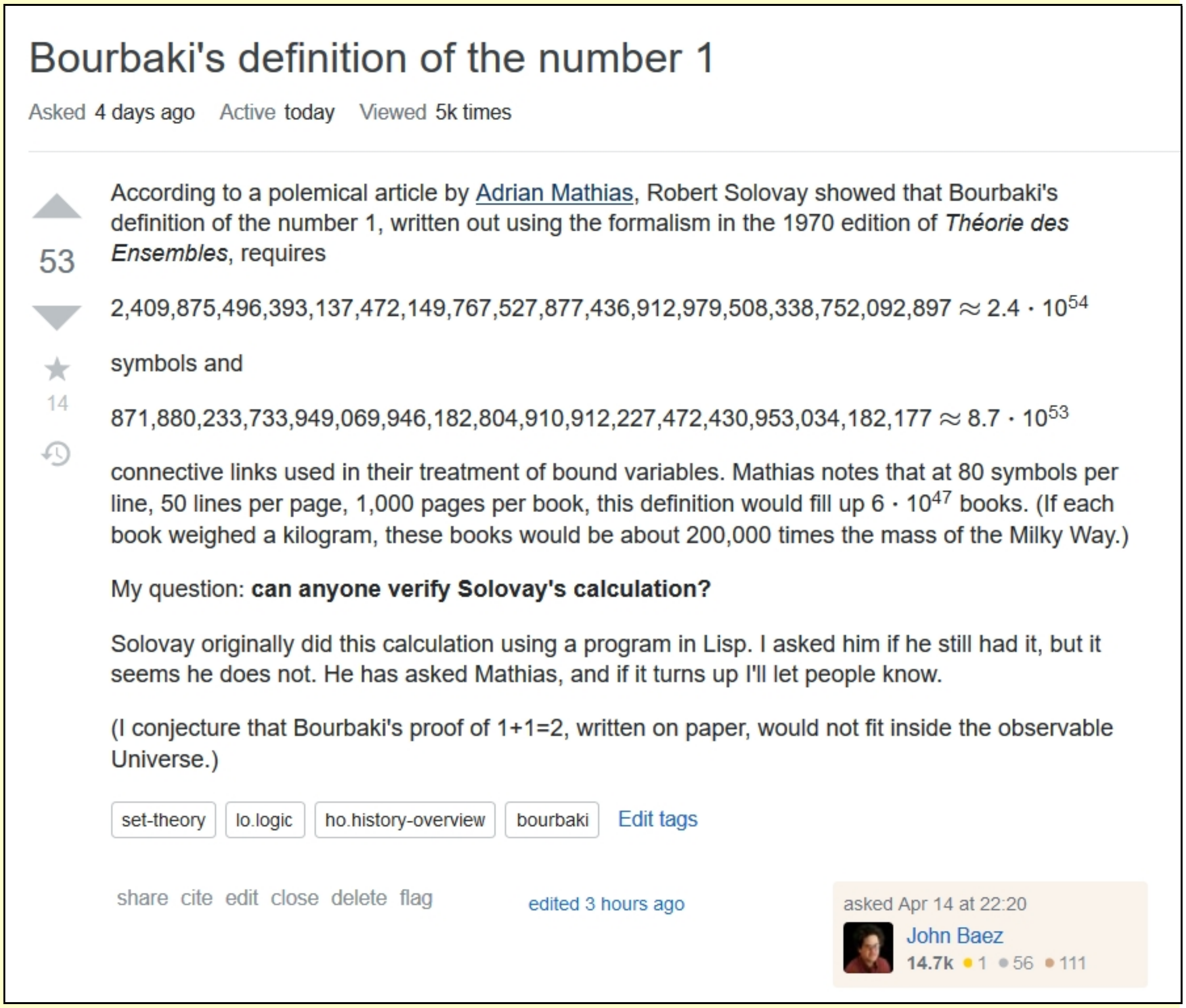
April 16, 2020

Bourbaki's final perfected definition of the number 1, printed out on paper, would be 200,000 times as massive as the Milky Way.

At least that's what a calculation by the logician Robert Solovay showed. But the details of that calculation are lost. So I asked around. I asked Robert Solovay, who is retired now, and he said he would redo the calculation.

5 https://www.dpmms.cam.ac.uk/~ardm/inefff.pdf
6 https://www.dpmms.cam.ac.uk/~ardm/inefff.pdf
7 https://www.dpmms.cam.ac.uk/~ardm/inefff.pdf
8 https://johncarlosbaez.wordpress.com/2020/04/13/bigness-part-1/

I asked on MathOverflow,[9] and was surprised to find my question harshly attacked. I was accused of "ranting". Someone said the style of my question was "awful".[10]



Bourbaki's definition of the number 1

Asked 4 days ago Active today Viewed 5k times

53

14

According to a polemical article by Adrian Mathias, Robert Solovay showed that Bourbaki's definition of the number 1, written out using the formalism in the 1970 edition of *Théorie des Ensembles*, requires

$$2{,}409{,}875{,}496{,}393{,}137{,}472{,}149{,}767{,}527{,}877{,}436{,}912{,}979{,}508{,}338{,}752{,}092{,}897 \approx 2.4 \cdot 10^{54}$$

symbols and

$$871{,}880{,}233{,}733{,}949{,}069{,}946{,}182{,}804{,}910{,}912{,}227{,}472{,}430{,}953{,}034{,}182{,}177 \approx 8.7 \cdot 10^{53}$$

connective links used in their treatment of bound variables. Mathias notes that at 80 symbols per line, 50 lines per page, 1,000 pages per book, this definition would fill up $6 \cdot 10^{47}$ books. (If each book weighed a kilogram, these books would be about 200,000 times the mass of the Milky Way.)

My question: **can anyone verify Solovay's calculation?**

Solovay originally did this calculation using a program in Lisp. I asked him if he still had it, but it seems he does not. He has asked Mathias, and if it turns up I'll let people know.

(I conjecture that Bourbaki's proof of 1+1=2, written on paper, would not fit inside the observable Universe.)

set-theory lo.logic ho.history-overview bourbaki Edit tags

share cite edit close delete flag edited 3 hours ago

asked Apr 14 at 22:20
John Baez
14.7k ● 1 ● 56 ● 111

Maybe they thought I was attacking Bourbaki. That's not my real goal here. I'm thinking of writing a book about large numbers, so I'm doing a bit of research.

Admittedly, I added the remark saying Mathias' paper is "polemical" after Todd Trimble, a moderator at MathOverflow, recommended doing some such thing.

Later Solovay said it would be hard to redo his calculation — and if he did he'd probably get a different answer, because there are different ways to make the definition precise.

But here's some good news. José Grimm redid the calculation. He did it *twice*, and got two different answers, both bigger than Solovay's. According to these results Bourbaki's definition of "1", written on paper, may be 400 billion times heavier than the Milky Way.[11]

[9] https://mathoverflow.net/questions/357498/bourbakis-definition-of-the-number-1
[10] https://mathoverflow.net/questions/357498/bourbakis-definition-of-the-number-1
[11] https://mathoverflow.net/a/357558/2893

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These calculations have been carried out by José Grimm; see [1] as well as [2]. According to one version of the formalism in the original Bourbaki, Grimm gets

$$16420314314806459564661629306079999627642979365493156625 \approx 1.6 \times 10^{55}$$

(see page 517 of [1, version 10]). The discrepancy with Solovay's number is probably due to some subtle difference of interpretation of some detail. Note that the English translation of Bourbaki introduces some "small" changes and Grimm gets a rather different value:

$$5733067044017980337582376403672241161543539419681476659296689 \approx 5.7 \times 10^{60}$$

EDIT: As suggested in the comments, here are the full citations for Grimm's papers.

[1] José Grimm. Implementation of Bourbaki's Elements of Mathematics in Coq: Part Two; Ordered Sets, Cardinals, Integers. [Research Report] RR-7150, Inria Sophia Antipolis; INRIA. 2018, pp.826. inria-00440786v10. doi:10.6092/issn.1972-5787/4771

[2] Grimm, J. (2010). Implementation of Bourbaki's Elements of Mathematics in Coq: Part One, Theory of Sets. Journal of Formalized Reasoning, 3(1), 79-126. doi:10.6092/issn.1972-5787/1899

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edited 2 days ago
John Baez
14.7k ● 1 ● 56 ● 111

answered Apr 15 at 15:19
Timothy Chow
46.9k ● 15 ● 205 ● 366

I'm now quite convinced that a full proof of 1 + 1 = 2 in Bourbaki's formalism, written on paper, would require more atoms than available in the observable Universe.

Of course, they weren't aiming for efficiency.

References

[1] John Baez Diary, April 14-16, 2020 (http://math.ucr.edu/home/baez/diary/april_2020.html)

[2] Halmos, Paul R., "Nicolas Bourbaki," (May 1957) in *Mathematics in the Modern World, Readings from Scientific American, with an introduction by Morris Kline*, W. H. Freeman, San Francisco, 1968. pp.77-81.

[3] "Nicolas Bourbaki," *Wikipedia* (https://en.wikipedia.org/wiki/Nicolas_Bourbaki, retrieved 4/27/2020)

[4] Bourbaki, Nicholas, "The Architecture of Mathematics," *The American Mathematical Monthly,* Vol. 57, No.4. (Apr., 1950), pp. 221-232. (http://links.jstor.org/sici?sici=0002-9890%28195004%2957%3A4%3C221%3AT AOM%3E2.0.CO%3B2-S)