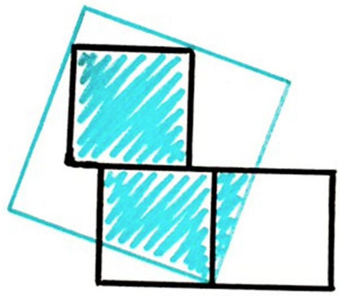
Geometric Puzzle Munificence

5 March 2020

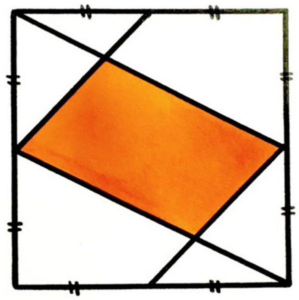
Jim Stevenson

#4 Three Little Squares Went Out to Play[1]



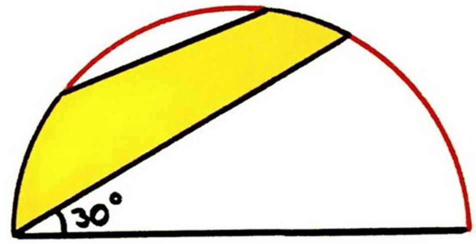
The three small squares each have an area of 4. What is the total shaded area?

#3 Parallelogram of Eternal Balance[2]



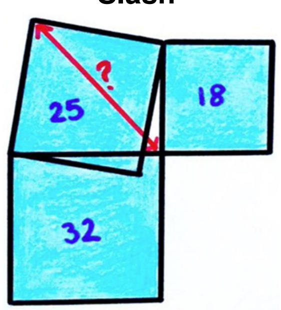
What fraction is shaded?

#2 Sunny Dome[3]



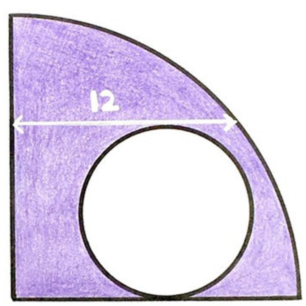
The two red arcs are the same length. What fraction of the semicircle is shaded?

#1 Three Squares and a Slash[4]



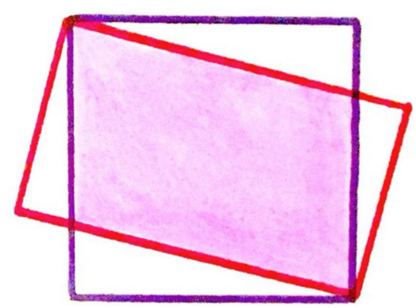
The areas of the squares are given. How long is the red line?

Bonus: #2 Hole in Pie[5]



What's the total shaded area?

Bonus: #1 Tipsy Rectangle[6]

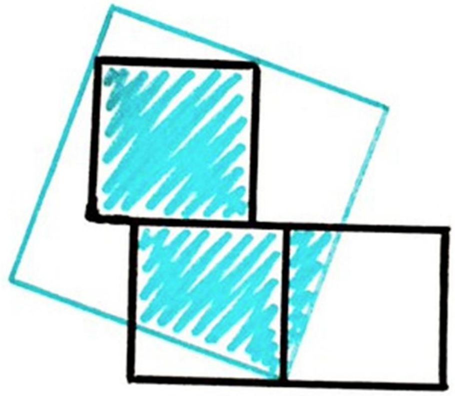


75% of the purple square is shaded. What percentage of the red rectangle is shaded?

Having fallen under the spell of Catriona Shearer's geometric puzzles again, I thought I would present the latest group assembled by Ben Orlin, which he dubs "Felt Tip Geometry",[7] along with a bonus of two more recent ones that caught my fancy as being fine examples of Shearer's laconic style. Orlin added his own names to the four he assembled and I added the names to my two, again ordered from easier to harder.

[1] https://twitter.com/Cshearer41/status/1216308670931271680, 12 January 2020
[2] https://twitter.com/Cshearer41/status/1212329908254457861, 1 January 2020
[3] https://twitter.com/Cshearer41/status/1223010993409875968, 30 January 2020
[4] https://twitter.com/Cshearer41/status/1213375792807370752, 4 January 2020
[5] https://twitter.com/Cshearer41/status/1229701889488539648, 18 February 2020
[6] https://twitter.com/Cshearer41/status/1228335812812693507, 14 February 2020
[7] https://mathwithbaddrawings.com/2020/02/12/felt-tip-geometry/, 12 February 2020

#4 Three Little Squares Went Out to Play



The three small squares each have an area of 4. What is the total shaded area?

My Solution

This is definitely easy to solve (Figure 1). The idea is to rotate 90º the right-most bottom small square and see that it coincides with the left-most bottom small square, with the triangle of shaded area in the right-most square overlaying the empty triangle in the left-most square.

Thus the total area of the shaded portion of the large square that is inside the three small squares is two small squares, or $2 \cdot 4 = 8$.

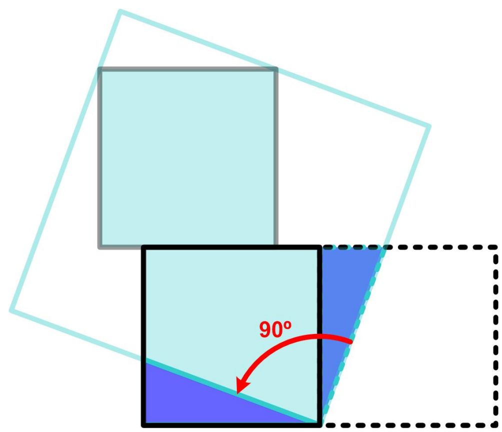
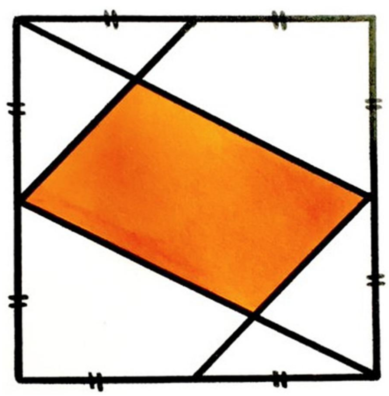


Figure 1 Problem Solution

#3 Parallelogram of Eternal Balance



What fraction is shaded?

My Solution

This puzzle is a bit more challenging. There may be a purely plane geometry solution, but I had to resort to analytic geometry.

If the containing square has unit sides, then the fraction of the square that is in the shaded area A, is just the area of A. The diagonal of the parallelogram becomes the base of the a triangle which is half the desired area, or $A/2$ (Figure 2). The base is the same length 1 as the side of the square, so we only need to find the altitude.

I found the equations for the lines bounding the desired area and then found their intersection point. Set the origin (0, 0) at the mid-point of the left edge of the square. Since the blue line goes from mid-point to mid-point of the edges of the square, its equation is $y = x$. The green line starts at a y-intercept of ½ and traverses 1 unit downwards until it crosses the horizontal axis. Therefore, its equation is $y = -½\, x + ½$. The intersection yields $^3/_2 x = {}^1/_2$ or $x = {}^1/_3$. Therefore, $y = {}^1/_3$ also, which is the altitude of the desired triangle. Therefore $^1/_2 A = {}^1/_2 \cdot 1 \cdot {}^1/_3 \Rightarrow A = {}^1/_3$.

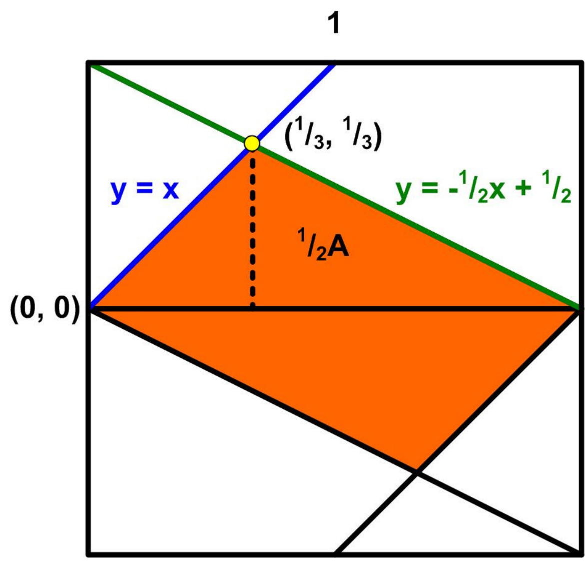
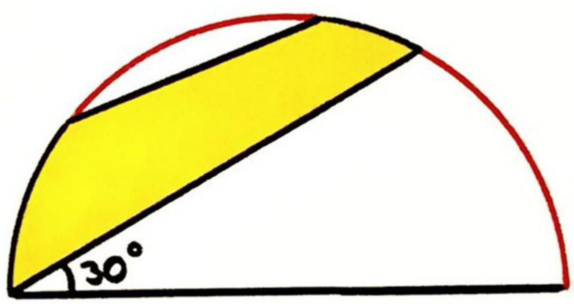


Figure 2 Problem Solution

#2 Sunny Dome



The two red arcs are the same length. What fraction of the semicircle is shaded?

My Solution

This puzzle is even more challenging and probably has a number of solutions. The first thing I did was slide the separate left-hand red arc over to join the right-hand red arc of the same size (Figure 3 and Figure 4).

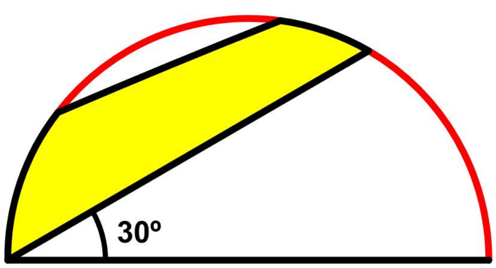


Figure 3 Initial Problem Setting

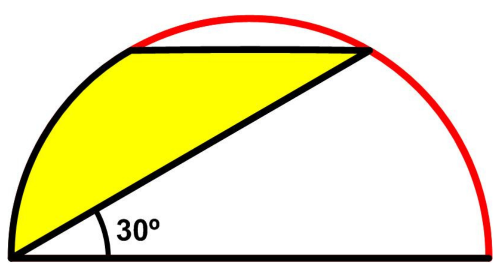


Figure 4 Solution Step 1

Now both arcs subtend the same central angle, namely twice the 30º angle or 60º (Figure 5). Since the blue dashed lines are also the radii of the semi-circle, they, together with the horizontal black line, form an equilateral triangle (Figure 6). This means the angle at the upper left vertex of the triangle is also 60º. The supplementary angle of the angles along the diameter of the semi-circle is 180 º – (60º + 60º) = 60º as well (Figure 6).

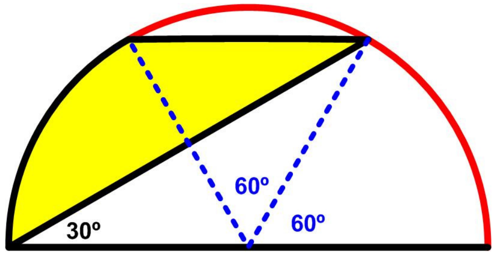


Figure 5 Solution Step 2

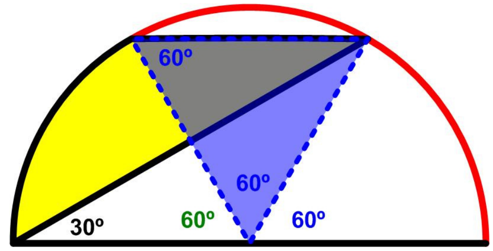


Figure 6 Solution Step 3

The complement of 30º + 60º is 90º, so the blue dashed line is perpendicular to the slanted black edge of the shaded region, and the complement of 90º + 60º is 30º (Figure 7). Since the sides of the blue equilateral triangle are the same as the radii, the two similar green right triangles in Figure 7 are congruent. Therefore, the original (yellow) shaded area is equal to pie segment shown in Figure 8, which is a third of the semi-circle.

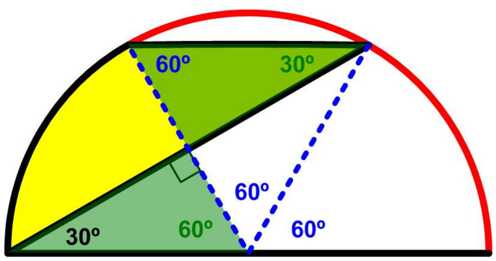


Figure 7 Solution Step 4

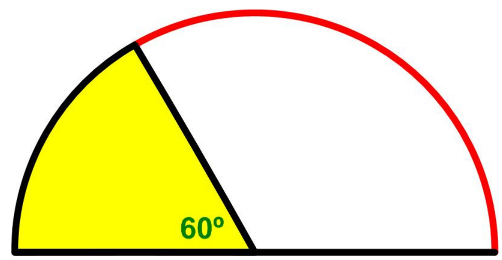
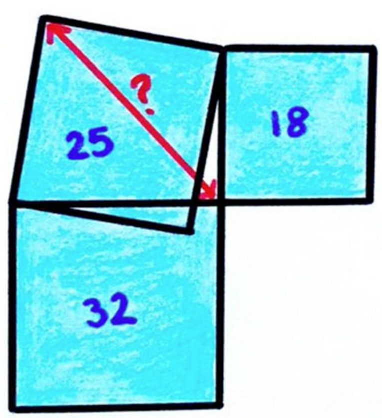


Figure 8 Shaded Area = $^1/_3$ Semi-circle

#1 Three Squares and a Slash



The areas of the squares are given. How long is the red line?

My Solution

This is another puzzle with a somewhat complicated solution. First, inscribe the rotated, area-25-square in a (yellow) square (Figure 9). Given the statement of the problem, this yellow square is the same as the 32 area square and so has an edge of length 4√2. Given the 90º rotational symmetries of the yellow circumscribed square and the area-25-square, the yellow right triangles showing in the figure are all congruent with each other (Figure 9).

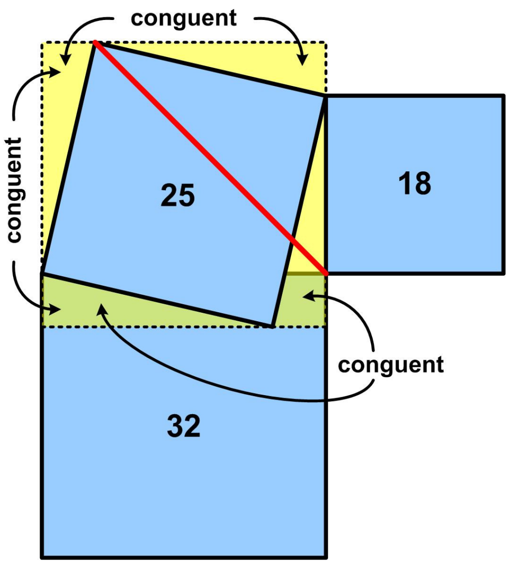


Figure 9 Solution Step 1

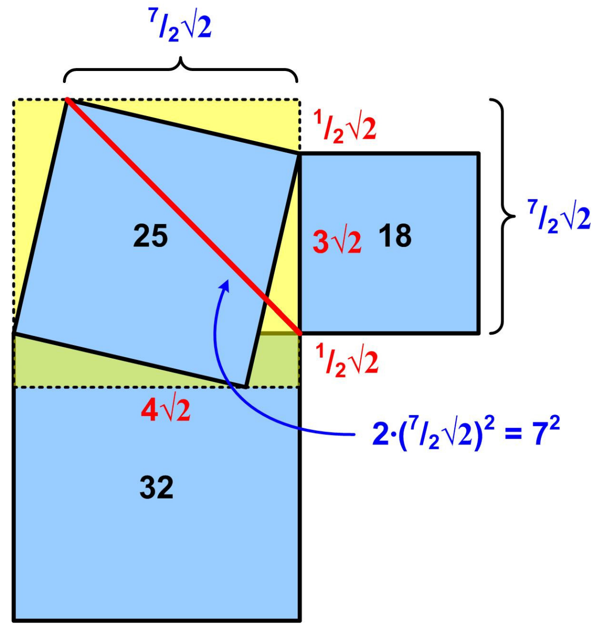


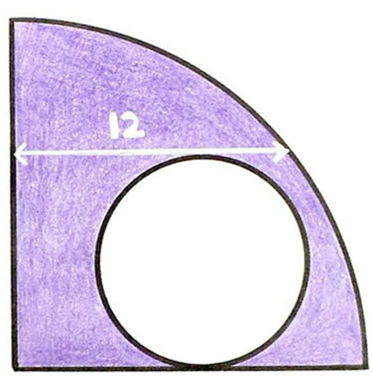
Figure 10 Final Solution Step

This means the area-18-square lies along the middle of the edge of the yellow square with equal yellow triangle segments on either side. Since the length of the edge of the yellow (area 32) triangle is $4\sqrt{2}$ and the edge of the 18-area-square is $3\sqrt{2}$, the equal lengths on either side must be $^1/_2(4\sqrt{2} - 3\sqrt{2}) = {^1/_2}\sqrt{2}$ (Figure 10). Therefore the length of the edge of the yellow square from its vertex to the red line is

$$3\sqrt{2} + {^1/_2}\sqrt{2} = {^7/_2}\sqrt{2}$$

Given the 90º rotational symmetry of the area-25-square and yellow square, the other edge must also be $^7/_2\sqrt{2}$ long. Therefore, the red line, being the hypotenuse of an isosceles right triangle must by $\sqrt{2}$ times the leg, or $\sqrt{2}\cdot{^7/_2}\sqrt{2} = 7$.

Bonus: #2 Hole in Pie



What's the total shaded area?

My Solution

Now back to an easier problem. If we annotate the problem as shown in Figure 11, then we have

$$R^2 = (2r)^2 + 12^2$$

so

$$R^2/4 = r^2 + 6^2$$

Therefore the shaded area is

$$^1/_4\pi R^2 - \pi r^2 = \pi 6^2 = 36\pi$$

Neat.

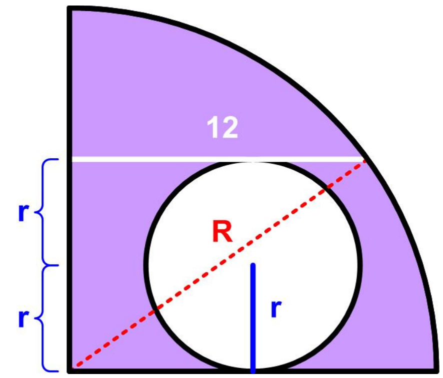
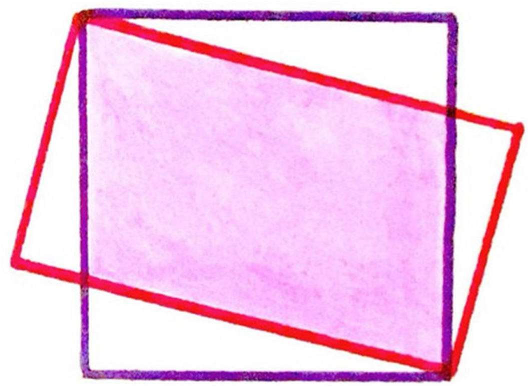


Figure 11 Problem Solution

Bonus: #1 Tipsy Rectangle



75% of the purple square is shaded. What percentage of the red rectangle is shaded?

My Solution

This is rather tricky, and especially opaque at first. Again, there are probably multiple solutions but this is what I thought of. I first noticed that the square and rectangle had a common diagonal and that all the vertices had right angles. That made me think of right triangles in a semi-circle with common hypotenuse equal to the diameter of the semi-circle. So Figure 12 shows the square and rectangle inscribed in a circle of radius 1. Given the 180º rotational symmetry about the center of the circle, we only need to consider the relationship of areas in the upper semi-circle. This reduces the problem to looking at areas of triangles. Having a radius of 1 means the area of the upper triangle in the purple square is 1 (Figure 13). Therefore, the *fraction* of area of the rectangle in the square is just

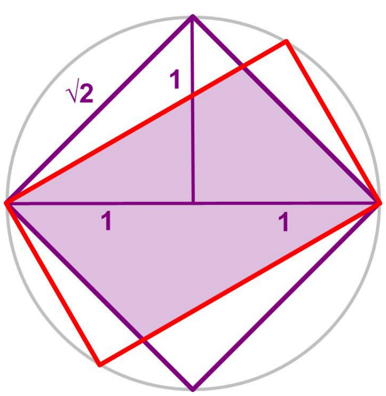


Figure 12

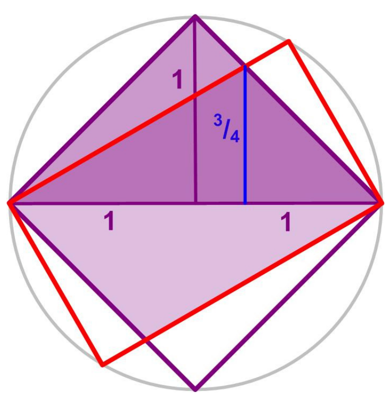


Figure 13

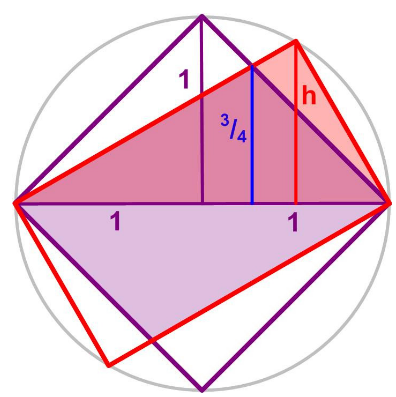


Figure 14

given by the area of the corresponding upper triangle, which we will call the "rectangle" triangle.

Furthermore, since all the triangles will have base equal to the diameter of the semi-circle, which is 2, and the areas of the triangles are all ½ base x altitude, all the areas of the triangles reduce to the values of the altitudes. Therefore saying the ratio of the area of the rectangle inside the square to the area of the square is $^3/_4$, is equivalent to saying the altitude of the "rectangle" triangle is $^3/_4$ (Figure 13).

So now we want to find the full area of the rectangle, or actually the one half of the area in the upper triangle (Figure 14), or equivalently the altitude h of this triangle. Then the ratio of the area of the rectangle in the square to the area of the rectangle will be $^3/_4 / h$.

This is where things start to get complicated and we resort to trigonometry. Let θ be the angle shown in Figure 15. Since the purple square produces 45º right triangles, the blue $^3/_4$ altitude for the "rectangle" triangle is also $^3/_4$ from the right-hand endpoint of the diameter, and so $^5/_4$ from the left-hand endpoint of the diameter. Therefore, $\tan\theta = {^3/_5}$. Now draw the (black) radius to the top of the altitude h (Figure 16). This defines the central angle for θ and therefore is equal to 2θ. We introduce the variable x for the distance to the altitude h. Then $\tan 2\theta = h/x$ and $1 = x^2 + h^2$.

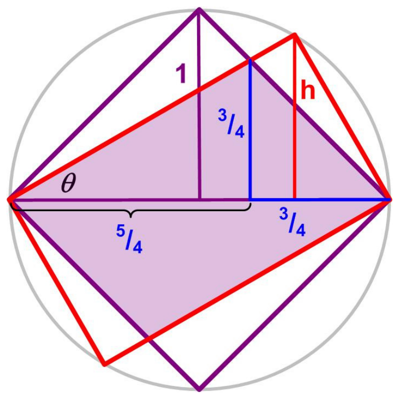


Figure 15

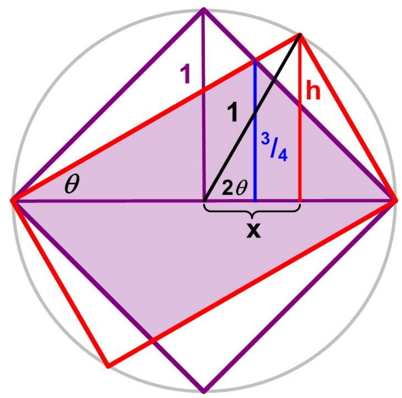


Figure 16

Using the double angle trigonometry identity for the tangent, we have

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \quad \Rightarrow \quad \frac{h}{x} = \frac{2\cdot\frac{3}{5}}{1-\frac{9}{25}}$$

Therefore, $x = {^8/_{15}}h$, and so

$$1 = (^8/_{15})^2\, h^2 + h^2$$

Thus

$$h = {^{15}/_{17}}$$

and so

$$\text{ratio of areas} = {^3/_4} / h = {^{17}/_{20}}$$