# Railroad Tie Problem

12 February 2020

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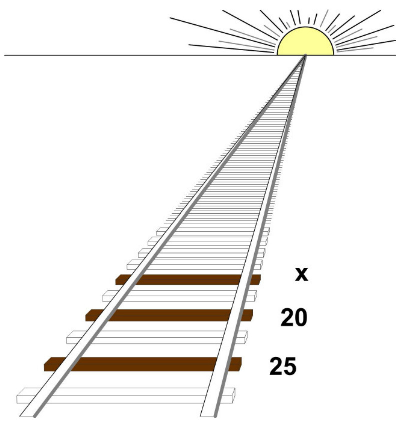


This is a surprisingly challenging puzzle from the *Mathematics 2020* calendar ([1] February).

The sketch is of equally spaced railroad ties drawn in a one point perspective. Two of the ties are perceived to the eye to be 25 feet and 20 feet respectively. What is the perceived length $x$ of the third tie?

Even though the ties are equally-spaced and of equal length in reality, from the point of view of perspective they are successively closer together and diminishing in length. The trick is to figure out what that compression factor is. I had to review my post on the Perspective Map[1] to get some clues.

## Solution

According to the principles of the perspective map, the diagonals through the opposite corners of the equal rectangles made by the ties and the rails are all parallel and will converge on a different vanishing point on the same horizon (Figure 1). This second set of converging lines determines the perceived spacing of the ties.[2]

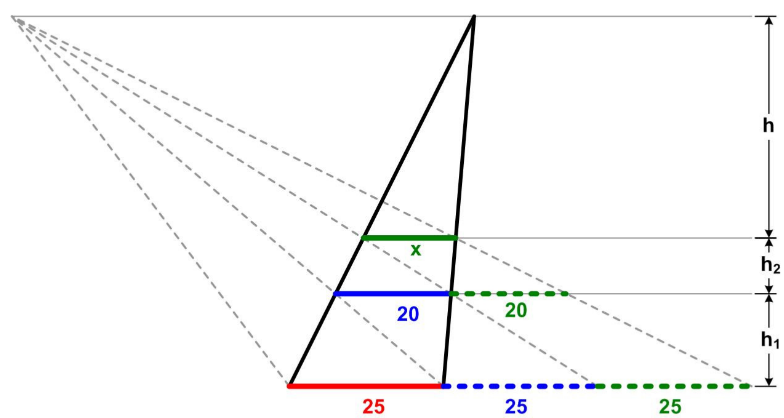


**Figure 1 Problem Solution**

So we have the following from the similar triangles determined by the right-most dashed line to the second vanishing point and the right-most solid line to the first vanishing point:

$$\frac{h_2}{20} = \frac{h_2 + h_1}{50} \quad \Rightarrow \quad h_1 = \frac{3}{2} h_2 \tag{1}$$

In addition, all the solid-lined triangles are similar, so

$$\frac{h}{x} = \frac{h + h_2}{20} = \frac{h + h_2 + h_1}{25} \tag{2}$$

The second equation in equations (2), together with equation (1), yields

[1] http://josmfs.net/2019/01/02/perspective-map/

[2] In the Perspective Map posting, check the solution to the *Futility Closet* problem given at the end of the post. It uses essentially the same procedure.

$$\frac{5}{4}(h+h_2)=h+\frac{5}{2}h_2 \quad \Rightarrow \quad \frac{1}{5}h=h_2 \tag{3}$$

The first equation in equations (2), together with equation (3), then yields

$$20h=(h+h_2)x=\frac{6}{5}hx \quad \Rightarrow \quad \boxed{x=\frac{5}{6}20=16\tfrac{2}{3}}$$

So the answer is that tie $x$ is perceived to be 16 feet 8 inches.

**Comment**. The nature of the *Mathematics 2020* calendar is such that all the answers are supposedly integral values, namely, the date on which the problem is given. Strangely, I made a mistake initially (I had $h_2/20 = h_1/25$ for equation (1)) and came out with the answer of 16 corresponding to the February date. On subsequent review I corrected the mistake and arrived at the fractional answer above. I recall some other answers in the calendar that do not come out as integers, but those problems say to round the result. In this case rounding would give the wrong answer. At this point I do not know the source of the discrepancy.

One thought is that this problem is slightly different from the Perspective Map set-up, since I am not assuming from the beginning that I have a grid of parallel equally-shaped rectangles. So I want to make sure that the triangle base segments in Figure 1 above are all equal to 25 feet.

**Shear Invariance**. Specifically, this problem follows the more general steps shown in Figure 2. First, we have the black triangles with a vanishing point on the (green) horizon line. Second, we draw a (red) line from a second vanishing point on the horizon line through the left endpoint of the thick bar down to the base line of the large black triangle. (In the actual Railroad Tie Problem this red line ran along a diagonal of the parallelogram. I wanted to treat the more general case.) Third, we draw a (blue) line from the second vanishing point to the other end of the thick black line. Finally, fourth, we extend the blue line to the base line of the large black triangle and connect its endpoint with the end point of the red line. I want to show the two base segments of the large triangles are equal.

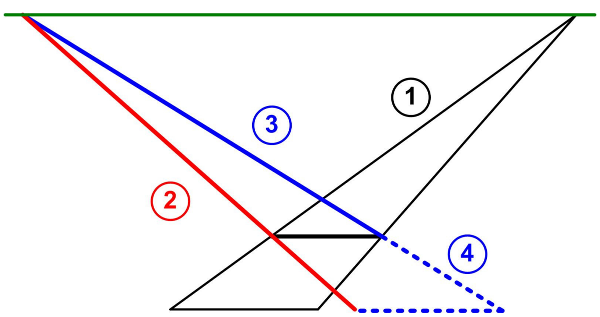


**Figure 2 Problem Set-up**

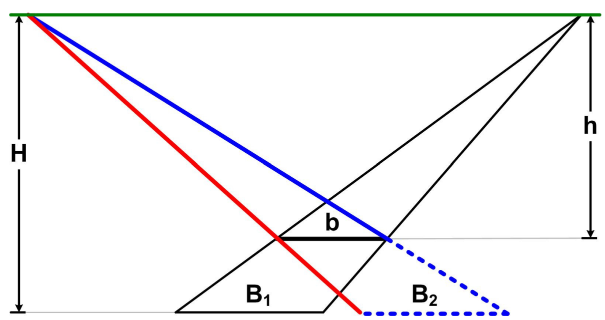


**Figure 3 Shear Invariance**

Figure 3 shows an annotated version of Figure 2. I use virtually the same shear-invariance argument as I used in the Three Coffin Problems[3] posting. Namely, via similar triangles, we have

$$\frac{b}{B_1}=\frac{h}{H}=\frac{b}{B_2}$$

so $B_1 = B_2$, which is what we wanted to show.

So I still am at a loss as to the source of the discrepancy between my fractional answer to the problem and the *Mathematics 2020* calendar integer solution.

[3] http://josmfs.net/2019/01/19/three-coffin-problems/

## References

[1] Rapoport, Rebecca and Dean Chung, *Mathematics 2020: Your Daily epsilon of Math*, Point Rock, Quarto Publishing Group, New York, 2020