Pinwheel Area Problem

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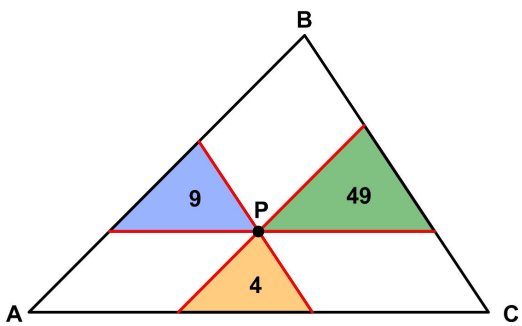


Here is another engaging problem from Presh Talwalkar ([1]).

Triangle Area 1984 AIME

Point *P* is in the interior of triangle *ABC*, and the lines through *P* are parallel to the sides of *ABC*. The three triangles shown in the diagram have areas of 4, 9, and 49. What is the area of triangle *ABC*?

The American Invitational Mathematics Exam (AIME) is the second qualifying test for selection in the US Mathematical Olympiad team. The test is 3 hours long with 15 questions (an average of 12 min/problem), and no calculators are allowed. It’s a challenging exam, and even the talented students who take it only average (mean and median) a score of 3—a mere 20 percent. Today’s puzzle is problem 3 from the 1984 test. I also want to thank a viewer in Sweden who told me a very similar problem was asked in a competition there for 14 year olds. It is not an outdated problem!

My Solution

Talwalkar assumes you remember that the ratio of the areas of two similar triangles is equal to the ratio of the squares of any corresponding sides of the two triangles.[1]

Consider Figure 1 where the small similar triangles in the original problem have been slid to one edge of the large similar triangle to show that the altitude and base of the large triangle is made up of the sums of the altitudes and bases of the small triangles. (Because the red lines through P are all parallel to the sides of the large triangle, all the triangles have the same angles and so are similar. Furthermore, the white spaces in the original figure are parallelograms and so the small triangles can be slid as shown in Figure 1.) Then the area A of the large triangle is given by

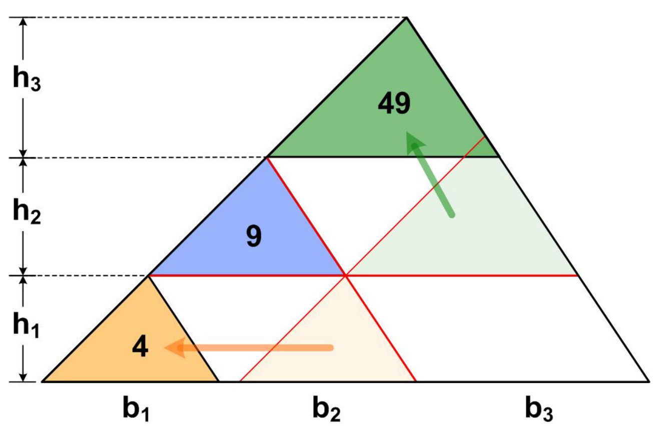


Figure 1 Problem Solution

$$A = \tfrac{1}{2}(b_1 + b_2 + b_3)(h_1 + h_2 + h_3)$$

We now employ the ratio of areas fact to give

$$\frac{4}{b_1^{\,2}} = \frac{9}{b_2^{\,2}} = \frac{49}{b_3^{\,2}}$$

[1] If triangle A is similar to triangle B, then there is a constant scale factor k such that if s_A and s_B are corresponding line segments on A and B respectively, then $s_A = k\, s_B$. The line segments could be sides or even altitudes of the triangles. Then area $\Delta A = k^2$ area ΔB. From this the statement follows (LTR).

and

$$\frac{4}{h_1^{\ 2}} = \frac{9}{h_2^{\ 2}} = \frac{49}{h_3^{\ 2}}$$

These results imply $b_1/b_2 = 2/3$ and $b_2/b_3 = 3/7$, or

$$b_2 = \frac{3}{7}b_3 \text{ and } b_1 = \frac{2}{7}b_3 .$$

And similarly,

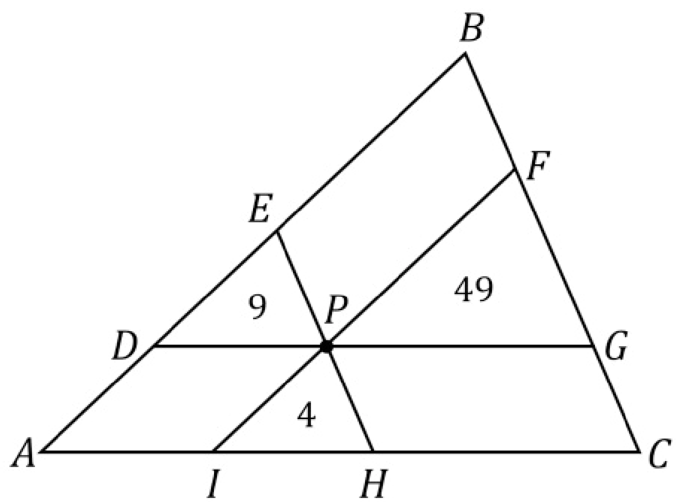
$$h_2 = \frac{3}{7}h_3 \text{ and } h_1 = \frac{2}{7}h_3 .$$

Therefore

$$A = \frac{1}{2}\left(\frac{2}{7}+\frac{3}{7}+1\right)b_3\left(\frac{2}{7}+\frac{3}{7}+1\right)h_3 = \left(\frac{2+3+7}{7}\right)^2 49 = (2+3+7)^2 = 144$$

Talwalkar's Solution

Label the diagram as follows. Because the lines through P are parallel to the sides of ABC, each of the three small triangles HIP, DEP, FGP is similar to ABC and so they are all similar to each other.



A triangle's area is proportional to its side length squared, so the square root of its area is proportional to its side length. We are given the information:

areas HIP:DEP:FGP = 4:9:49

Because the triangles are similar, their sides will have a ratio equal to the square root of their areas.

sides HIP:DEP:FGP = 2:3:7

Suppose $HI = 2x$. Because the triangles are similar, $DP = 3x$ and $PG = 7x$.

Now notice $ADPI$ is a parallelogram. This is because lines through P are parallel to the sides of ABC, so IP and AD are parallel and so are DP and AI. Opposite sides in a parallelogram have equal length, so $AI = DP = 3x$. Similarly $HPGC$ is a parallelogram, and $HC = PG = 7x$.

Consequently we can calculate AC has a length:

$$AC = AI + IH + HC = 3x + 2x + 7x = 12x$$

Triangles ABC and HIP are similar, and their sides are in a ratio:

$$AC:IH = (12x):(2x) = 6:1$$

Consequently their areas will be in a ratio of the square of their sides:

area $AC:IH = 6^2:1^2 = 36:1$

Thus triangle *ABC* has an area 36 times as large as *HIP*, so its area is:

$$36(4) = 144$$

References

I was emailed a similar problem from a student in Sweden.

1. Art of Problem Solving 1984 AIME Problem 3
https://artofproblemsolving.com/wiki/index.php/1984_AIME_Problems/Problem_3
2. Web 2.0 Calc forum similar question
https://web2.0calc.com/questions/another-one-of-my-triangle-questions
3. Math 7210 solution at the University of Georgia
http://jwilson.coe.uga.edu/MATH7200/ProblemSet4.3.html

References

[1] Presh Talwalkar, "Triangle Area 1984 AIME," *Mind Your Decisions*. January 9, 2020 (https://mindyourdecisions.com/blog/2020/01/09/triangle-area-1984-aime)