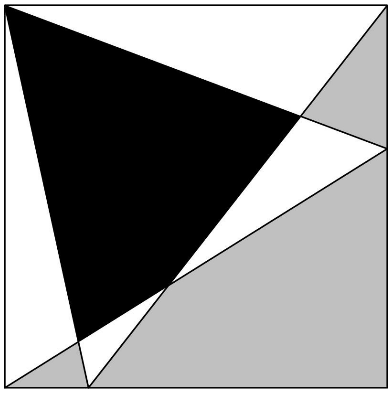
Modern Art

17 June 2015

Jim Stevenson



This is a problem from a while back at *Futility Closet* ([1]).

Which part of this square has the greater area, the black part or the gray part?

My Solution

First we show that the black part in the figure is made up of the overlap of two triangles. We color them blue and green. Since the base and altitudes of both triangles are the edges of the square, their area is one half the area of the square. Denote this area by $\mathbf{T}$ and the area of the square by $\mathbf{S}$. Then $\mathbf{T} = ½\ \mathbf{S}$. Let $\mathbf{T_1}$ represent the are of the blue triangle and $\mathbf{T_2}$ the area of the green triangle. Then

$$\mathbf{T_1} = \mathbf{A_1} + \mathbf{A_2} + \mathbf{A_3} = ½\,\mathbf{S} \quad \text{and}$$

$$\mathbf{T_2} = \mathbf{B_1} + \mathbf{B_2} + \mathbf{B_3} = ½\,\mathbf{S}$$

and so $\mathbf{T_1} + \mathbf{T_2} = \mathbf{S}$. But the colored region in the square from the overlapping triangles counts the overlap area (original black area) twice. So the actual area of the colored region is $\mathbf{S} - \mathbf{A_2}$. This means the region in the square outside the overlapping colored triangles (the gray area) has area $\mathbf{A_2}$. But that is also the area of the overlap (black area in original figure). Thus we have shown the black region has the same area as the gray region in the original figure.

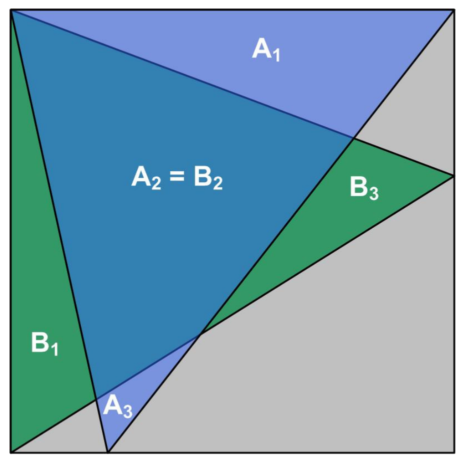


Figure 1 My Solution

Futility Closet Solution

The Futility Closet solution is essentially equivalent to mine.

Color the indicated regions dark gray:

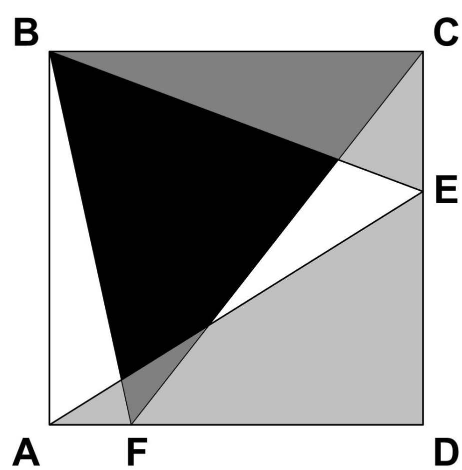


Figure 2 Futility Closet Solution

Now if we add the dark gray regions to the black region, we get triangle BCF, whose area is half that of the whole square (a triangle's area is one-half base times height). If we add the dark gray areas to the light gray areas, we again produce a region that's half the area of the square (because it's the full square minus triangle ABE). So the black part and the light gray part are equal.

Using our notation, the dark gray areas are $\mathbf{A}_1$ and $\mathbf{A}_3$, and the black area is $\mathbf{A}_2$. So adding these three we get the area $\mathbf{T}_1 = \frac{1}{2}\,\mathbf{S}$. But adding $\mathbf{A}_1$ and $\mathbf{A}_3$ to the light gray area again yields the area of half the square. That is,

$$\mathbf{A}_1 + \text{(light gray area)} + \mathbf{A}_3 = \tfrac{1}{2}\,\mathbf{S} = \mathbf{A}_1 + \text{(black area)} + \mathbf{A}_3$$

So subtracting $\mathbf{A}_1 + \mathbf{A}_3$ from both sides yields (light gray area) = (black area).]

By V. Proizvolov, from the Russian science magazine Kvant.

References

[1] "Modern Art" *Futility Closet*, 13 May 2015, (http://www.futilitycloset.com/2015/05/13/modern-art/, retrieved 6/17/15)