Mysterious Dopplegänger Problem

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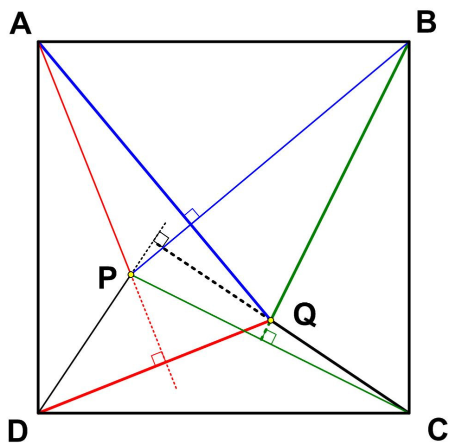

I found this problem from the Math Challenges section of the 2002 *Pi in the Sky* Canadian math magazine for high school students ([1]) to be truly astonishing.

Problem 4. Inside of the square *ABCD*, take any point *P*. Prove that the perpendiculars from *A* on *BP*, from *B* on *CP*, from *C* on *DP*, and from *D* on *AP* are concurrent (i.e. they meet at one point).

How could such a complicated arrangement produce such an amazing result? I didn't know where to begin to try to prove it. My wandering path to discovery produced one of my most satisfying "aha!" moments.

Solution

I verified that the solution I found was the same as given in the next issue of *Pi in the Sky* ([2]). My technique for solving geometric problems is to plot them using Visio in Microsoft's Office. The application allows for very precise positioning of lines, circles, ellipses, and angles, along with proportional shrinking and the rigid motions of rotation, reflection, and translation. I used Visio to produce the figure above. Watching the perpendiculars all intersect at a common point as I constructed the figure fed my astonishment. So how in the world to proceed?

To unclutter the figure somewhat, I eliminated the blue lines from A and B (Figure 1). Since Visio works on a grid, I was able to find the center of the square and draw lines from the center to the original point P and intersection point Q. I noticed that the lines were the same length and at right angles to each other. That looked promising. Then I noticed that the lengths of each pair of solid lines of the same color from the corners of the square to P and Q were the same length. That seemed even more promising.

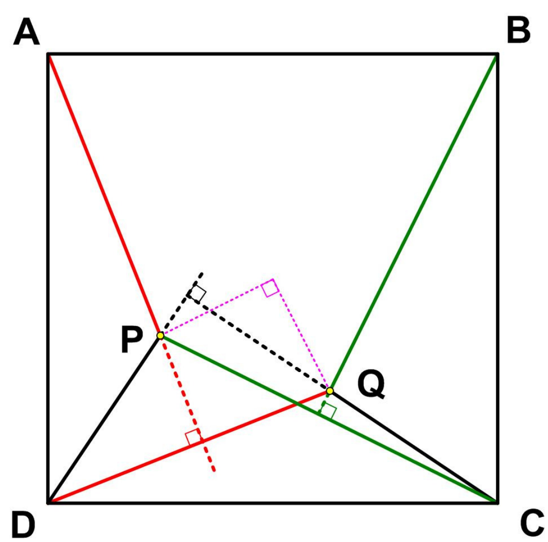

Figure 1 Exploration and Observation

And then the solution hit me—the "aha!" moment. The perpendicular lines to the point Q were the 90° rotated image of the original four lines to the point P (Figure 2)! So of course they would intersect at a common point, since they had originally.

Rarely does one find such an elegant, one-step solution to a seemingly complicated and impenetrable problem. I had never seen this problem before, but I think it should take its place among the best.

[1] Revision: I had plotted the original figure incorrectly. Fortunately, the original solution idea still worked, only now the figure was rotated 90º counterclockwise instead of 90º clockwise.

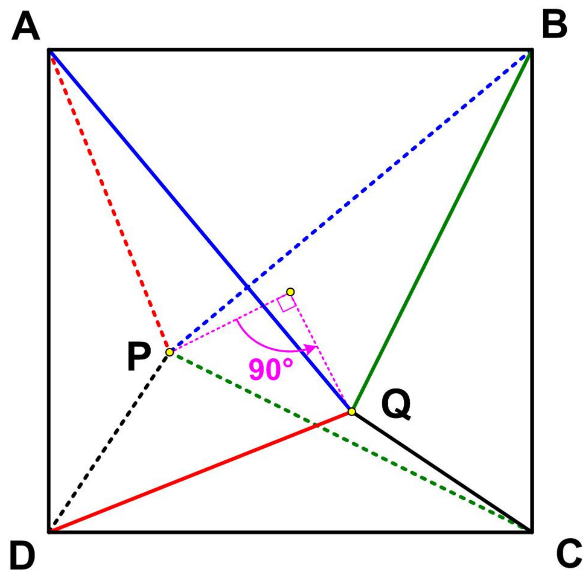

Figure 2   Solution

References

[1] "Math Challenges," *Pi in the Sky*, Issue 5, September 2002

[2] "Math Challenges," *Pi in the Sky*, Issue 6, March 2003