Geometric Puzzle Madness

15 December 2019

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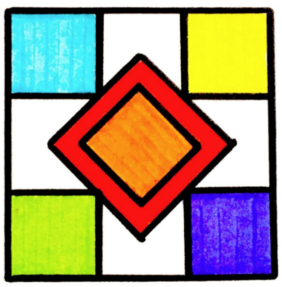
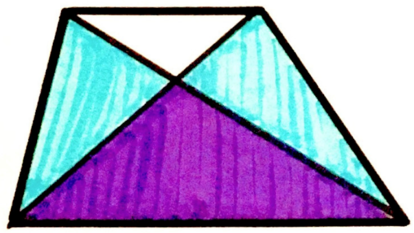
#3 Sci-Fi Stained Glass



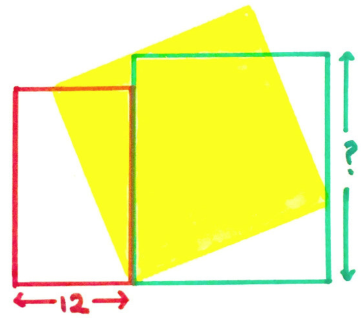
Pattern of squares. All 6 colored areas are equal. What fraction is shaded?

#2 Kiddle's Riddle



Total blue area equals purple area. What fraction of trapezium is shaded?

#1 House of Square Sun

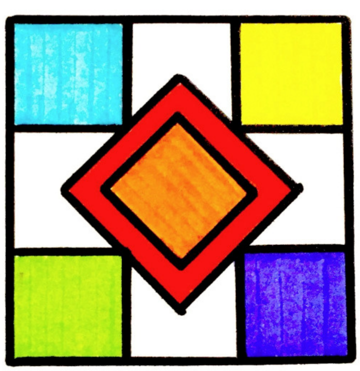


Area of yellow square is twice area of red rectangle. What's height of green rectangle?

I have been subverted again by a recent post by Ben Orlin, "Geometry Puzzles for a Winter's Day,"[1] which is another collection of Catriona Shearer's geometric puzzles, this time her favorites for the month of November 2019 (which Orlin seems to have named himself). I often visit Orlin's blog, "Math with Bad Drawings",[2] so it is hard to kick my addiction to Shearer's puzzles if he keeps presenting these intriguing collections. Her production volume is amazing, especially as she is able to maintain the quality that makes her problems so special.

The Stained Glass puzzle generated some discussion about needed constraints to ensure a solution. Essentially, it was agreed to make explicit that *the drawing had vertical and horizontal symmetry in the shapes*, that is, flipping it horizontally or vertically kept the same shapes, though some of the colors might be swapped. This appears evident from the drawing (and I had assumed it), which raises an issue about the laconic style of Shearer's puzzles: when can you assume evident aspects of her drawings and when can you not? I assumed the orientation of the centered squares was horizontally and vertically symmetric. What I thought needed proving was how big was the red square.

Puzzle #3: Sci-Fi Stained Glass



A pattern of squares. All six colored areas are equal. What fraction is shaded?

(Assuming horizontal and vertical symmetry of the shapes)

[1] https://mathwithbaddrawings.com/2019/12/09/geometry-puzzles-for-a-winters-day/
[2] https://mathwithbaddrawings.com/

My Solution

The horizontal and vertical symmetry of the figure implies the colored squares in the corners of the containing square have their interior vertices on the corresponding midpoints of the edges of the central red square.

Now to consider the size of the red square. First, we rotate the central orange square 45° and assume its corners intersect the large red square at the midpoints of its edges (Figure 1). By drawing all the diagonals of both squares we divide the visible areas of the squares into the same number of congruent isosceles right triangles, thus demonstrating that the visible area of the red and orange squares is the same. Notice that if the rotated orange square does not intersect the edges of the red square, then the red square's visible area is larger than that of the orange square. And if the corners of the orange square fall outside the edges of the red square, then the red square's visible area is smaller than that of the orange square.

Since the orange square intersects the red square edges at the latter's midpoints, then the squares form an equally-spaced grid as shown in Figure 2. Therefore, we can shift the colored pieces about to arrive at Figure 3, which shows that the colored areas add up to 2/3 the area of the enclosing square.

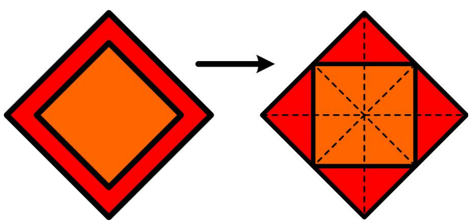


Figure 1 Equal Areas

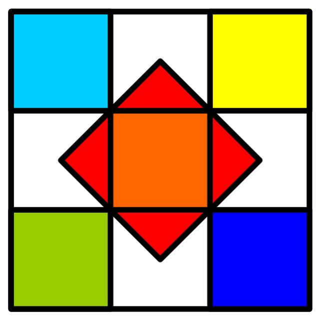


Figure 2 Regular Grid

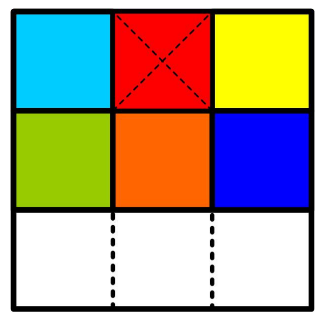
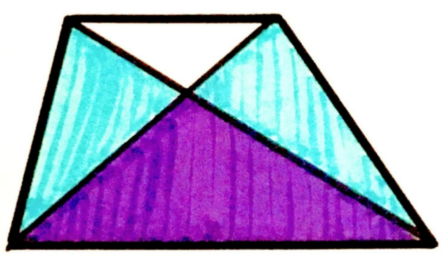


Figure 3 Final Solution

Puzzle #2: Kiddle's Riddle



The total blue area is equal to the purple area.
What fraction of the trapezium is shaded?

My Solution

We begin by labeling the original figure as shown in Figure 4 where the capital letters represent the areas of the corresponding triangles. Then

$$A + C = A + D \Rightarrow C = D$$

since the triangles for A + C and A + D have the same altitude and base. We are given that

$$A = C + D$$

Therefore, C = ½ A and the area of the colored region is 2A.

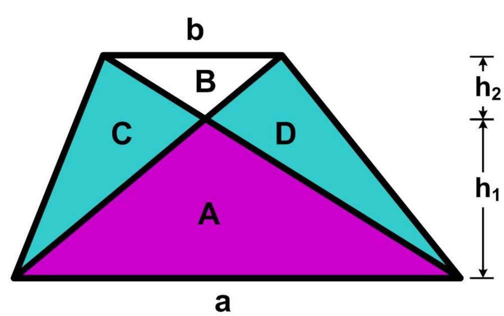


Figure 4 Figure Parmeterized

Now we need to characterize the area B. Since the triangles with areas A and B are similar (LTR), the ratio of their areas is as the squares of their bases or the squares of their altitudes:

$$\frac{B}{A} = \frac{{h_2}^2}{{h_1}^2}$$

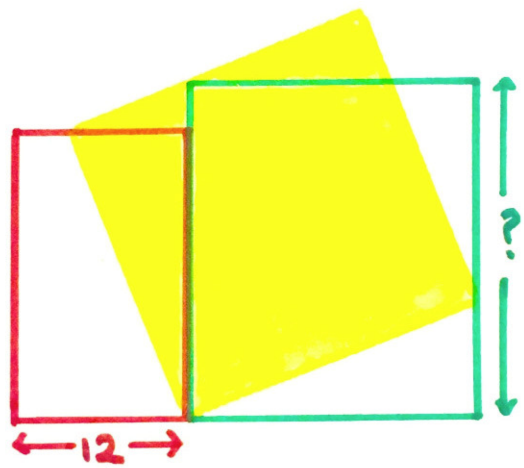
But $$A + C = ½\,(h_1 + h_2)\,a \;\Rightarrow\; C = ½\,h_2\,a = ½\,A = ¼\,h_1\,a$$

which means $$\frac{h_2}{h_1} = \frac{1}{2} \;\Rightarrow\; B = ¼\,A$$

Thus the fraction of the trapezoid that is colored is

$$\boxed{\frac{A+C+D}{A+C+D+B} = \frac{2A}{2A+\frac{1}{4}A} = \frac{8}{9}}$$

Puzzle #1: The House of the Square Sun



The yellow square has double the area of the red rectangle. How tall is the green rectangle?

My Solution

I was fooled at first into thinking the green rectangle was a square. But that leads to a contradiction. The diagram is full of similar right triangles. But then the simple solution popped out.

Label the diagram as in Figure 5. Then the statement that the area of the yellow square is twice the area of the red rectangle becomes

$$x^2 = 2{\cdot}12\,z = 24\,z.$$

One can then proceed via similar triangles or the equivalent trigonometric statement

$$\cos\theta = \frac{z}{x} = \frac{x}{y}$$

But that implies

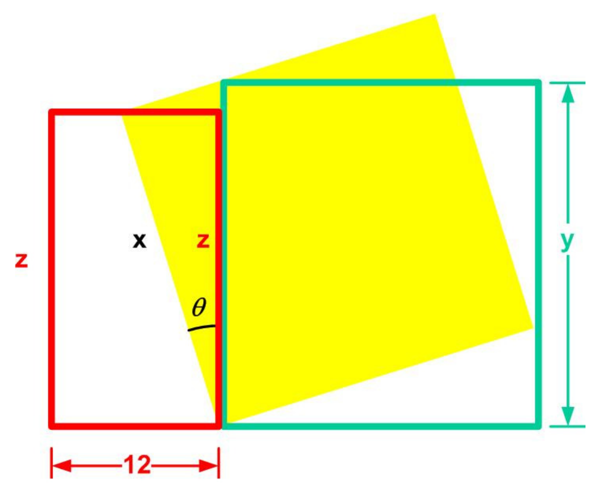


Figure 5 Solution

$$\boxed{y = \frac{x^2}{z} = \frac{24z}{z} = 24}$$

Commentary

Since I don't have a Twitter account, I was not able to view the comments on Catriona Shearer's Twitter account to see what hers or others' solutions were. Olin had said about the House of the Sun problem, "I found this one the trickiest of the three. Indeed, Catriona says that these three are 'in ascending order of difficulty (in my mind, at least).' She also explains, tantalizingly, 'Unless I missed it, I don't think anyone posted the solution I originally came up with. It's unusual for me to feel like my method is unique!' " I am certainly curious about her solution.