Circle Tangent Chord Problem

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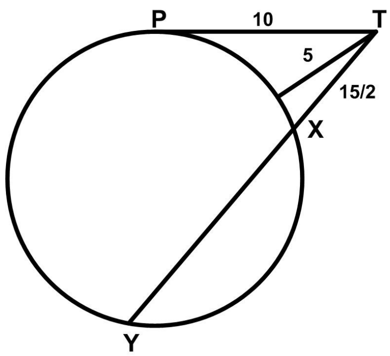

This is another problem from the Math Challenges section of the 2000 *Pi in the Sky* Canadian math magazine for high school students ([1]).

Problem 4. From a point P on the circumference of a circle, a distance PT of 10 meters is laid out along the tangent. The shortest distance from T to the circle is 5 meters. A straight line is drawn through T cutting the circle at X and Y. The length of TX is 15/2 meters.

(a) Determine the radius of the circle,

(b) Determine the length of XY.

My Solution

First, we need to show that **the minimum distance from *T* to the circle is along a radial line through the center of the circle at *O*.** In Figure 1 the grey circular arcs around T represent constant distance contours from T. As the circles shrink towards T, the two points of intersection on both the green circle and grey coalesce towards one point, the tangent point on the two circles. This means the two circles share a tangent line a that point. But the radius of each circle is perpendicular to the tangent line, and so they are parallel, and in fact collinear, since they share a point.

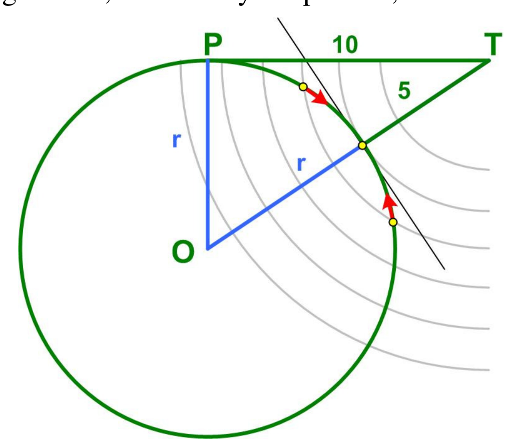

Figure 1 Minimum Distance to Circle

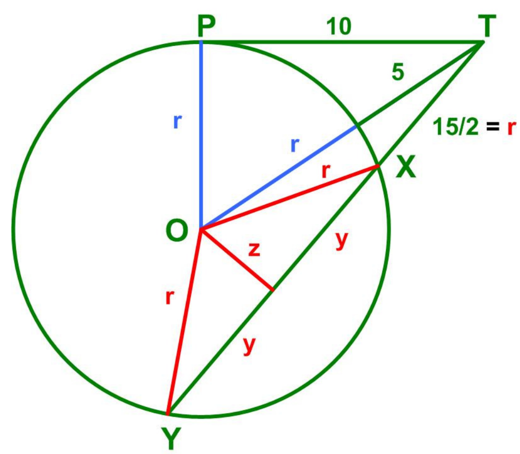

Figure 2 Solution

Therefore we have a right triangle so that

$$r^2 + 10^2 = (r + 5)^2$$

or $10r = 75$ or $r = 15/2$, which is the same distance as TX.

As shown in Figure 2, draw the radii from the center of the green circle O to X and Y, and drop the perpendicular of length z from O onto the segment XY. Since OXY is an isosceles triangle, the perpendicular also bisects XY. Label the two halves of the segment XY with length y ($2y = XY$). Then we get two equations from the Pythagorean Theorem:

$$y^2 + z^2 = r^2 \quad \text{and} \quad z^2 + (y + \mathrm{r})^2 = (r + 5)^2$$

Eliminating x gives $$(r^2 - y^2) + (y^2 + 2yr + r^2) = (r^2 + 10r + 25)$$

or $$2y = \frac{10r + 25 - r^2}{r} \tag{1}$$

Substituting $r = 15/2$ yields $$XY = 2y = 35/6$$

Pi in the Sky **Solution**

Their solution ([2]) is somewhat similar, but they did not prove that the shortest distance from T to the circle was along the radial line of the circle. Moreover, I have some problems with their solution. I have reproduced it with some minor edits for clarity and the addition of figure references. (By the way, no figures were supplied in the original problem statement, only in the solution.)

We have that $r^2 + 10^2 = (r + 5)^2$ (Figure 3). So $r = 15/2$ and so therefore $r = TX$ also. The length x of XY can be determined from the equations (Figure 4)

$$2\cos^2\alpha - 1 = \cos 2\alpha = \frac{x}{2r} \tag{2}$$

that is, $$x = \frac{r^2 + 10r + 25}{r} = \frac{35}{6} \tag{3}$$

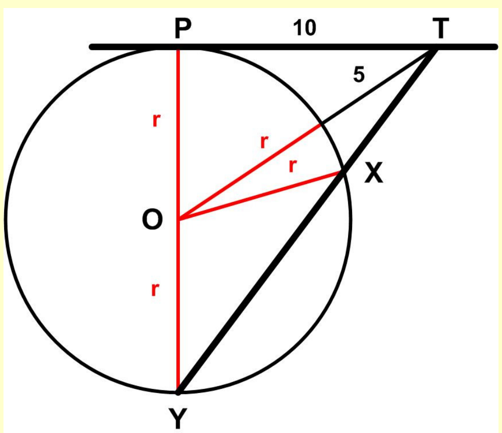

Figure 3 Solution Step1

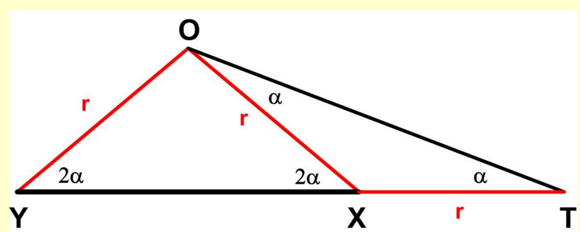

Figure 4 Solution Step2

Comment. I don't understand this solution. First, their Figure 3 shows the point Y falling diametrically opposite P so that PY becomes a diameter and PXY becomes a right triangle. Why? This was not proved. It is not immediately obvious and I didn't assume it, as you can see from my Figure 2.

(Actually, from Figure 5, $y^2 + z^2 = r^2$ means $(2y)^2 + (2z)^2 = (2r)^2$ or by similar triangles, the figure does imply $P'XY$ is a right triangle, but not PXY. That is, there is no requirement that $P = P'$.)

Second, given that PXY is a right triangle, then equation (2) makes sense, but how does it imply equation (3)? Again, it is not obvious. In fact,

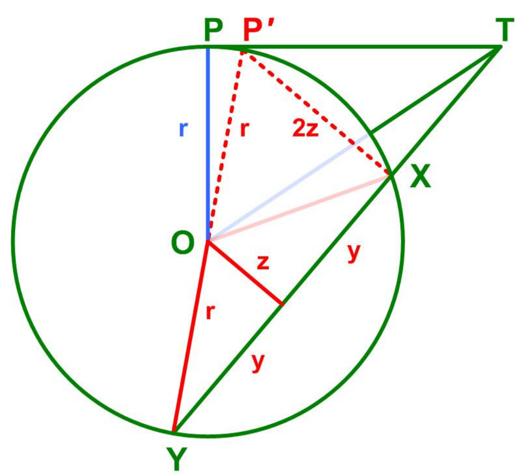

Figure 5 Solution Error

equation (3) is wrong: it should be $-r^2$ in the numerator (see equation (1)). We can assume this last is a typo, but the derivation still is not immediately clear (to me).

References

[1] "Math Challenges," *Pi in the Sky*, Issue 1, June 2000

[2] "Math Challenges," *Pi in the Sky*, Issue 2, December 2000