Tipsy 3-4-5 Triangle

26 August 2019

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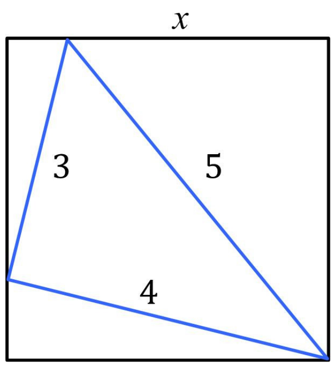

Presh Talwalkar had another interesting problem.[1]

A triangle is drawn inside a square with sides 4, 3, and 5, as shown. What is the length of the square's side?

The problem looks simple at first, but it takes some care to avoid some hideous quartic equations.

My Solution

I proceeded in a somewhat pedestrian fashion. I labeled the figure as shown in Figure 1. At first I tried a bunch of equations with the Pythagorean theorem, but that led to the nasty quartics. Then I decided to use the areas of the square and inscribed triangles, from which I produced the ugly equation

$$x^2 = \frac{1}{2}\left[3\cdot 4 + x\sqrt{4^2 - x^2} + x\sqrt{5^2 - x^2} + \left(x - \sqrt{4^2 - x^2}\right)\left(x - \sqrt{5^2 - x^2}\right)\right]$$

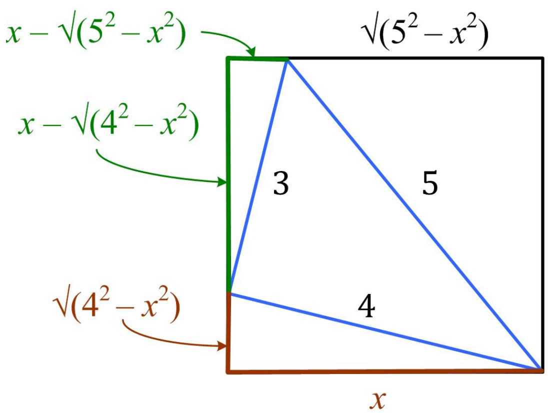

Figure 1 Solution Setup

Multiplying the last two terms and canceling a lot of radical terms fortunately leads to

$$2x^2 = 12 + x^2 + \sqrt{(4^2 - x^2)(5^2 - x^2)}$$

$$x^2 - 12 = \sqrt{(4^2 - x^2)(5^2 - x^2)}$$

Squaring both sides yields

$$x^4 - 24x^2 + 3^2 4^2 = 5^2 4^2 - (5^2 + 4^2)x^2 + x^4$$

$$(16 + 25 - 24)x^2 = 5^2 4^2 - 3^2 4^2 = 4^2 4^2$$

$$17x^2 = 16^2$$

$$x = 16/\sqrt{17}$$

Talwalkar's Solution

Talwalkar used a less computationally intensive approach, but with more geometric reasoning.

There are often many ways to solve such problems. I will present a method that I felt was direct but also instructive.

First, recall that a 3-4-5 triangle is a special right triangle, so the angle between the 3 and 4 sides is a 90 degree right angle (Figure 2). Now ignore the side of 5, and we have two triangles whose hypotenuses are 4 and 3. We will show they are similar triangles. Suppose one angle in the triangle of hypotenuse 4 has an angle *a*, so its other acute angle is 90 – *a* degrees (Figure 3). In the triangle of hypotenuse 3, we can calculate the lower left angle as follows. The side of the square is a straight line with measure 180 degrees, so the angle in the triangle has to be 180 – 90 – (90 – *a*) = *a*. This means

[1] Talwalkar, Presh "What Is The Square's Side Length?" August 15, 2019 (https://mindyourdecisions.com/blog/2019/08/15/what-is-the-squares-side-length/#more-32410)

the other angle in the triangle is 90 – *a* degrees (Figure 4). As corresponding angles in these triangles are equal, they are similar triangles.

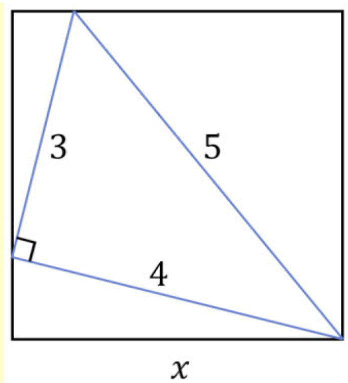

Figure 2

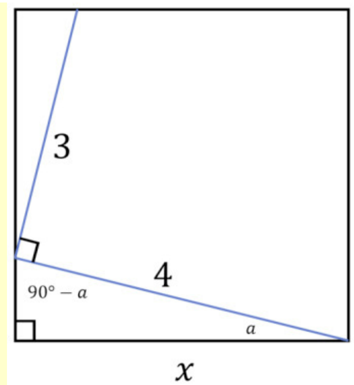

Figure 3

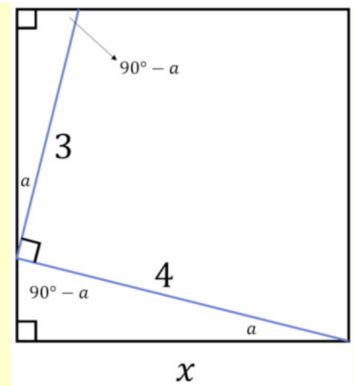

Figure 4

Now suppose the short leg of the triangle with hypotenuse 4 has a length of y. As the square side length is x, the remaining distance has to be $x - y$ (Figure 5). Since the triangles are similar, we can equate the ratio of the long leg to the hypotenuse, giving:

$$(x - y)/3 = x/4$$

$$4(x - y) = 3x$$

$$x = 4y$$

$$x/4 = y$$

We substitute this value into the diagram for y (Figure 6).

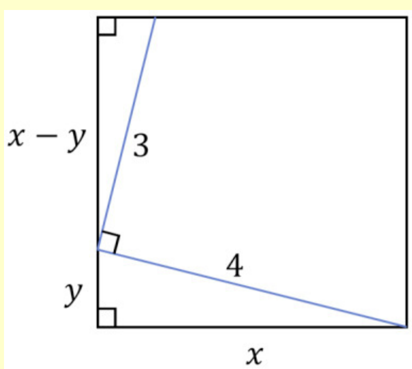

Figure 5

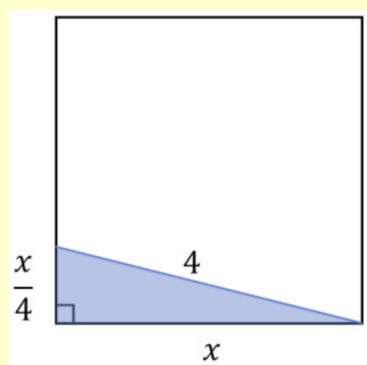

Figure 6

Finally we can use the distance formula to get:

$$x^2 + (x/4)^2 = 4^2 \Rightarrow (17/16)x^2 = 16$$

Since we want a positive value of x, we get the solution:

$$x = 16/\sqrt{17}$$

I enjoy how the problem is easy to state and its solution employs many principles of geometry.

Sources

- Instagram post of problem: https://www.instagram.com/p/Btw9ZfyBz5w/
- Math forum discussion of problem: http://mathhelpforum.com/geometry/236852-area-square-right-triangle-inside.html