Pairwise Products

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This 2005 four-star problem from Colin Hughes at *Maths Challenge*[1] is also a bit challenging.

Problem

For any set of real numbers, R = $\{x, y, z\}$, let sum of pairwise products,

$$S = xy + xz + yz.$$

Given that $x + y + z = 1$, prove that $S \leq 1/3$.

Again, I took a different approach from Maths Challenge, whose solution began with an unexplained premise.

My Solution

I saw the problem again as a constrained optimization problem where we are trying to find the maximum for S under a constraint on x, y, z and that the maximum is $\leq 1/3$. Again, this problem can typically be solved with Lagrange Multipliers (See for example [1] pp.859-861 or my previous post "Maximum Product"). We are trying to maximize the function S where

$$S(x, y, z) = xy + xz + yz \tag{1}$$

subject to the constraint

$$g(x, y, z) = x + y + z - 1 = 0. \tag{2}$$

The Lagrange multiplier approach says that at an extreme point, the gradients satisfy $\nabla f = \lambda \nabla g$ for some scalar λ, where λ is the Lagrange multiplier. We can assume the set of real numbers x, y, z under consideration are non-negative, so that at least $S \geq 0$ on that subset, and thus greater than any negative values. Therefore the gradients are both pointing in the positive x, y, z directions and so the extreme value is a maximum.

Let $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ represent the unit basis vectors for $\mathbf{R}^3$. Then

$$\nabla S = \frac{\partial S}{\partial x}\mathbf{i} + \frac{\partial S}{\partial y}\mathbf{j} + \frac{\partial S}{\partial z}\mathbf{k} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$$

and

$$\nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

So at the maximum point, $\nabla S = \lambda \nabla g$ implies we have the following three equations

$$y + z = \lambda$$

$$x + z = \lambda$$

$$x + y = \lambda$$

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[1] "Pairwise Products" Problem ID: 225 (24 May 2005) Difficulty: 4 Star at mathschallenge.net. "A four-star problem: A comprehensive knowledge of school mathematics and advanced mathematical tools will be required." (https://mathschallenge.net/problems/pdfs/mathschallenge_4_star.pdf)

Subtracting the first two yields $x = y$ and subtracting the last two yields $y = z$. Therefore

$$x = y = z = \lambda/2$$

at the maximum point. Substituting into equation (2) yields $\lambda = 2/3$, which means

$$x = y = z = 1/3.$$

So that the maximum value of S is

$$S(1/3, 1/3, 1/3) = 1/9 + 1/9 + 1/9 = 1/3.$$

This means for all x, y, z, $S(x, y, z) \leq 1/3$.

I find this solution elegant, simple, and slick. Now we get to compare it to the messy Maths Challenge solution, which of course avoids calculus.

Maths Challenge Solution

Let $x = 1/3 + a$, $y = 1/3 + b$, and $z = 1/3 + c$.[2] Therefore,

$$x + y + z = 1/3 + a + 1/3 + b + 1/3 + c = 1 + a + b + c.$$

But as $x + y + z = 1$, we deduce that $a + b + c = 0$. Then

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc) = 0$$

$$2(ab + ac + bc) = -(a^2 + b^2 + c^2)$$

$$ab + ac + bc = -(a^2 + b^2 + c^2)/2 = -d,$$

where $d \geq 0$.

So
$$\begin{aligned} xy + xz + yz &= (1/3 + a)(1/3 + b) + (1/3 + a)(1/3 + c) + (1/3 + b)(1/3 + c) \\ &= 1/9 + a/3 + b/3 + ab + 1/9 + a/3 + c/3 + ac + 1/9 + b/3 + c/3 + bc \\ &= 1/3 + (2/3)(a + b + c) + ab + ac + bc \end{aligned}$$

As $a + b + c = 0$ and $ab + ac + bc = -d$, we get,

$$S = xy + xz + yz = 1/3 - d \leq 1/3$$ **Q.E.D.**.

Wonderfully ugly arithmetic. No way could I negotiate this labyrinth without a lot of arithmetic errors. At last, an example that shows the virtues of calculus.

References

[1] Thomas Jr., George B. (late), Maurice D. Weir, Joel R. Hass, *Thomas' Calculus: Early Transcendentals* 13th Edition, Pearson, 1200 pp, 2014

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[2] JOS: Why? I suppose the problem solver is supposed to guess the maximum might be at (1/3, 1/3, 1/3) and then have to show any point away from this point will yield a lesser value for S.