Flipping Parabolas

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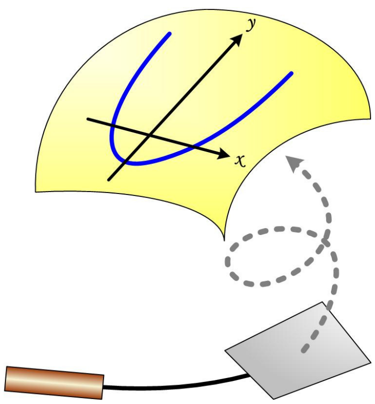

This is a stimulating problem from the UKMT Senior Math Challenge for 2017. The additional problem "for investigation" is particularly challenging. (I have edited the problem slightly for clarity.)

The parabola with equation $y = x^2$ is reflected about the line with equation $y = x + 2$. Which of the following is the equation of the reflected parabola?

A $x = y^2 + 4y + 2$ B $x = y^2 + 4y - 2$ C $x = y^2 - 4y + 2$

D $x = y^2 - 4y - 2$ E $x = y^2 + 2$

For investigation: Find the coordinates of the point that is obtained when the point with coordinates (x, y) is reflected about the line with equation $y = mx + b$.

Problem Solution

I knew how to reflect a figure about the 45º line, just swap the coordinates: $(x, y) \to (y, x)$. So the idea was to translate the figures so that the line $y = x + 2$ became $y' = x'$, flip the parabola about the 45º line, and translate the resulting figures back (Figure 1).

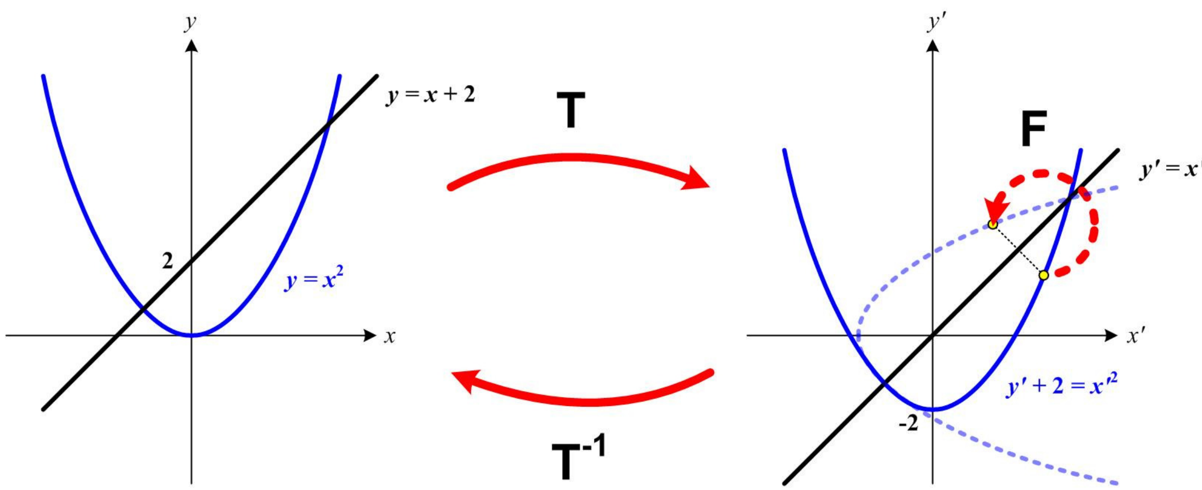

Figure 1 Transformations to Solve Problem

Define

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y-2 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad F : \begin{pmatrix} x' \\ y' \end{pmatrix} \to \begin{pmatrix} y' \\ x' \end{pmatrix}, \quad \text{and} \quad T^{-1} : \begin{pmatrix} x' \\ y' \end{pmatrix} \to \begin{pmatrix} x' \\ y'+2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then composing transformations, we have $F' = T^{-1} \circ F \circ T : (x, y) \to (y - 2, x + 2)$, that is,

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\mathbf{T}}{\to} \begin{pmatrix} x \\ y-2 \end{pmatrix} \overset{\mathbf{F}}{\to} \begin{pmatrix} y-2 \\ x \end{pmatrix} \overset{\mathbf{T^{-1}}}{\to} \begin{pmatrix} y-2 \\ x+2 \end{pmatrix}$$

So flipping $y = x^2$ about $y = x + 2$ yields

$$(x + 2) = (y - 2)^2 \text{ or } x = y^2 - 4y + 2 \text{ (Answer C)}$$

"For Investigation" Solution

I had never considered this before, so I found the question quite interesting.

Translation. The translation and its inverse were simple: T: $(x, y) \rightarrow (x, y - b)$ and T^{-1}: $(x, y) \rightarrow (x, y + b)$. The challenge was determining the flip F about the resulting line $y = mx$. I thought a vector approach might be easiest. Hopefully Seniors in today's high schools are augmenting their plane geometry with vector geometry, and even some linear algebra with linear transformations and matrices.

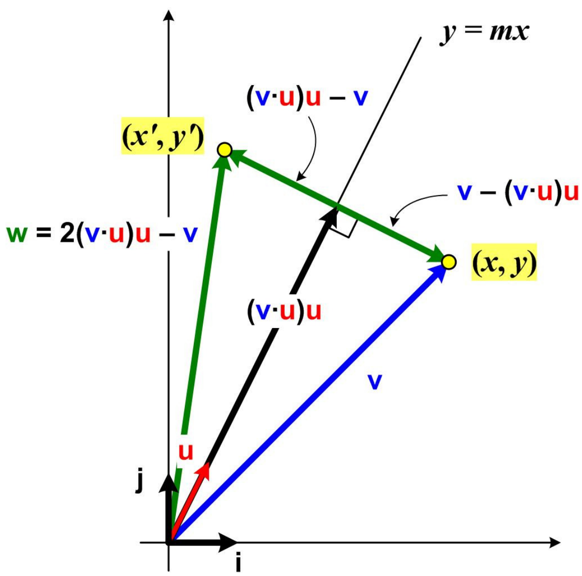

Figure 2 Flip About $y = mx$

Flip (Reflection). Figure 2 illustrates the idea. Choose a point (x, y) to be flipped, draw a vector **v** from the origin to (x, y), and then project **v** perpendicularly onto the line $y = mx$. Subtract the projection from **v** to get a vector perpendicular to $y = mx$ and of length the distance of (x, y) from the line $y = mx$. Take the negative of this vector and add it to the projection of **v** to get the vector **w** to the desired flipped point (x', y'). Unfortunately, carrying out the implied computations gets a bit hairy.

Let **u** be the unit vector along $y = mx$. Then

$$\mathbf{u} = \frac{1}{\sqrt{1+m^2}}(\mathbf{i} + m\,\mathbf{j})$$

and

$$\mathbf{v} = x\,\mathbf{i} + y\,\mathbf{j}$$

therefore, the projection of **v** onto **u** is given by

$$(\mathbf{v}\cdot\mathbf{u})\mathbf{u} = \frac{1}{1+m^2}\left[(x + my)\,\mathbf{i} + (mx + m^2 y)\,\mathbf{j} \right]$$

Then $\mathbf{w} = 2(\mathbf{v}\cdot\mathbf{u})\mathbf{u} - \mathbf{v}$ becomes

$$\mathbf{w} = \left[\frac{2(x+my)}{1+m^2} - x\right]\mathbf{i} + \left[\frac{2(mx+m^2y)}{1+m^2} - y\right]\mathbf{j}$$

or

$$\mathbf{w} = \frac{1}{1+m^2}\left[\left(2my - (m^2-1)x\right)\mathbf{i} + \left(2mx + (m^2-1)y\right)\mathbf{j} \right] \quad (1)$$

If $m = 1$, then $\mathbf{w} = y\,\mathbf{i} + x\,\mathbf{j}$, which is the original flip around the 45º line F: $(x, y) \rightarrow (y, x)$.

An easier way to see the general expression for F would be via matrices:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{1+m^2}\begin{pmatrix} -(m^2-1) & 2m \\ 2m & (m^2-1) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad (2)$$

Then for $m = 1$, F becomes

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$$

as before. If $m = 2$ and $(x, y) = (5, 5)$, then

$$F\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

So F: (5, 5) → (1, 7). (This is the example in Figure 2.)