Mathematics, And The Excellence Of The Life It Brings

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I am a regular reader of Ash Jogalekar's blog Curious Wavefunction, but I found my way to his latest via the eclectic website 3 Quarks Daily, also highly recommended. I could not resist the title, "Mathematics, And The Excellence Of The Life It Brings". The entirety of the post was about the mathematician Shing-Tung Yau's recent memoir, *The Shape of a Life* ([1]), but Jogalekar's introductory remarks about his personal involvement with mathematics stirred so many personal recollections of my own, that I thought I would provide an excerpt, followed by my own comments. Furthermore, he also addresses in passing the perennial question of whether math is invented or discovered.

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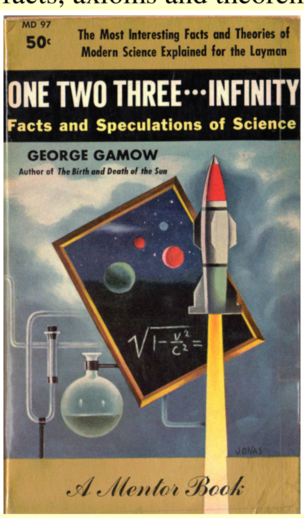
Mathematics, And The Excellence Of The Life It Brings

Ashutosh Jogalekar,[1] August 05, 2019

Mathematics and music have a pristine, otherworldly beauty that is very unlike that found in other human endeavors. Both of them seem to exhibit an internal structure, a unique concatenation of qualities that lives in a world of their own, independent of their creators. But mathematics might be so completely unique in this regard that its practitioners have seriously questioned whether mathematical facts, axioms and theorems may not simply exist on their own, simply waiting to be discovered rather than invented. Arthur Rubinstein and Andre Previn's performance of Chopin's second piano concerto sends unadulterated jolts of pleasure through my mind every time I listen to it, but I don't for a moment doubt that those notes would not exist were it not for the existence of Chopin, Rubinstein and Previn. I am not sure I could say the same about Euler's beautiful identity connecting three of the most fundamental constants in math and nature – e, pi and i. That succinct arrangement of symbols seems to simply be, waiting for Euler to chance upon it, the way a constellation of stars has waited for billions of years for an astronomer to find it.



My 1956 Copy

The beauty of music and mathematics is that anyone can catch a glimpse of this timelessness of ideas, and even someone untrained in these fields can appreciate the basics. The most shattering intellectual moment of my life was when, in my last year of high school, I read in George Gamow's "One, Two, Three, Infinity" ([2]) about the fact that different infinities can actually be compared. Until then the whole concept of

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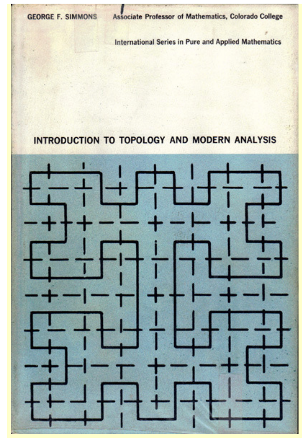
infinity had been a single concept to me, like the color red. The question of whether one infinity could be "larger" than another sounded as preposterous to me as whether one kind of red was better than another. But here was the story of an entire superstructure of infinities which could be compared, studied and taken apart, and whose very existence raised one of the most famous, and still unsolved, problems in math – the Continuum Hypothesis.[2] The day I read about this fact in Gamow's book, something changed in my mind; I got the feeling that some small combination of neuronal gears permanently shifted, altering forever a part of my perspective on the world.

Anyone who has seriously studied mathematics for any extended period of time also knows the complete immersion that can come with this study. In my second year of college I saw a copy of George F. Simmons's book "Introduction to Topology and Modern Analysis" ([3]) at the house of a mathematically gifted friend and asked to borrow it out of sheer curiosity. Until then mathematics had mainly been a matter of utilitarian value to me and most of my formal studies had been grounded in the kind of practical, calculus-based math that are required for solving problems in chemistry and physics. But Gamow's exposition of countable and uncountable infinities had whetted my mind for more abstract stuff. The greatest strength of Simmons's book is that it is entirely self-contained, starting with the bare basics of set theory and building up gradually. It's also marvelously succinct, almost austere in the brevity of its proofs.



My 1963 Copy

The book swept me off my feet, and the first time I started on it I worked through the theorems and problems right through the night; I can still see myself sitting at the table, the halo of a glaringly bright table lamp enclosing me in this special world of mathematical ideas, my grandmother sleeping outside this world in the small room that the two of us shared. The next night was not much different. After that I was seized by an intense desire to understand the fundamentals of topology – compactness, connectedness, metric and topological spaces, the Heine-Borel theorem, the whole works. Topology seemed to me like a cathedral – in fact the very word "spaces" as in "vector spaces" or "topological spaces" conjured up (and still do) an intricate, self-reinforcing cathedral of axioms, corollaries, lemmas and theorems resting on certain rules, each elegantly supporting the rest of it, being gradually built – or perhaps discovered – through the ages by its great practitioners, practitioners like Cantor, Riemann, Hilbert and Banach. It appeared like a great machine with perfectly enmeshed gears flawlessly fitting into each other and enabling great feats of mechanical efficiency and beauty. I was fortunate to find an enthusiastic professor who trained students for the mathematical olympiad, and he started spending several hours with me every week explaining proofs and helping me get over roadblocks. This was followed by many evenings of study and discussion, partly with a like-minded friend who had been inspired to get his own copy of Simmons's book. I kept up the routine for several months and got as far as the Stone-Weierstrass theorem before other engagements intruded on my time – I wasn't majoring in mathematics after all. But the intellectual experience had been memorable, unforgettable.

My Recollections

I, too, read Gamow's book in high school and his discussion of infinity was also my introduction to it as well. I was totally enthralled with his presentation, along with the wonderful story of the exponential growth of grains of wheat. Namely, as a seemingly modest reward for inventing chess, the grand vizier Sissa Ben Dahir asked King Shirham of India to supply him with grains of wheat on each square of a chess board with the first square having one grain, the second two grains, the third

[2] https://en.wikipedia.org/wiki/Continuum_hypothesis

four grains, and so on, each square having twice as many grains as its predecessor. After some calculations Gamow showed "the amount requested by the grand vizier was that of the *world's wheat production for the period of some two thousand years*!" Gamow covered other exciting topics for a high-schooler including relativity and cosmology. I was quite seduced.

But Jogalekar's reference to the Simmons topology book is even more remarkable, I too found it an amazing and wonderful book. I have to confess I found the book in my second year of graduate work and not the second year of my undergraduate studies. (If Jogalekar could read and understand the book in his sophomore year of college, then mathematics certainly lost a talented acolyte.) To truly understand the significance of my discovery of the book one has to understand the nature of graduate level math textbooks in the early 1960s—they were deplorable, mainly because they didn't exist. Those that did exist were often monographs on special subjects for the expert and not surveys for the novitiate, or they were in the original German, French, Polish, or Russian.

There were a few books available, and those that were, were often too expensive for a graduate student. We used Kelly's *General Topology* (1955) ([4]) which was quite abstract and axiomatic. That is, it followed the top-down abstract approach spawned by the Bourbaki[3] movement, where mathematical ideas have been generalized to their most abstract state and then presented in a rhythmic Definition-Theorem-Proof-Remark flow. I checked Kelly again and saw it had no examples, and the relatively few problems were relegated to the end of the chapters, and even then they were more of an abstract nature. This confirmed my impression of the book's abstractness and lack of concreteness.

Given the paucity of texts, we had to become scribes as we tried to keep up with lectures that spread hieroglyphics across the blackboard. We only hoped that the professor would not constantly stand in front of his writing, or worse yet, start erasing it with his left hand as he wrote with his right (these things happened). One particular lecture in algebraic topology was memorable for the incident when the professor stopped in the middle of a scribbled proof, pausing in consternation as to how to proceed. He turned his back on the class and hunched over the blackboard. We could see he was drawing a picture, but it was hidden from us. Suddenly he erased the picture and turned to declare "It's obvious". Well, the uproar that ensued was magnificent. "Do you mean to say there are diagrams we could see?" demanded the class. (This was a topology class after all.) But the professor dismissed it as a diversion from the authentic presentation.

So what does this have to do with Simmons? First, after giving an abstract definition, Simmons had lots of examples to illustrate more concretely the idea it captured. Sometimes he led with the examples to motivate a definition. Furthermore, he had lots of exercises that came at the end of individual sections rather than at the end of the chapter. These exercises further amplified the examples and also extended the ideas to more general views in anticipation of later abstractions. Finally, the text displayed helpful diagrams to clarify proofs and concepts.

But the most unusual aspect of Simmons's writing was its narrative flow—highly unusual in most math texts. I believe this was the quality that pulled Jogalekar into the book and kept him spellbound. I am not quite sure how Simmons did it, but perhaps it was the plentiful use of English where others might have used mathematical notation. The stenographic role we had been driven to in our lectures forced us to economize as much as possible in writing down what was said. Math is particularly helpful in this regard since it frequently uses a minimal set of phrases, such as, "for every", "there exists", "such that", "is an element of", "therefore", "if …, then…", etc. These and many more phrases can be reduced to symbols, thanks perhaps to the efforts of symbolic logic in the beginning of the 20th century. For example, the phrases just mentioned can be written: $\forall$, $\exists$, $\ni$, $\in$, $\therefore$, $\Rightarrow$. So my notes became dense and mind-numbing—often requiring a slower pace of decipherment. Simmons

3 https://en.wikipedia.org/wiki/Nicolas_Bourbaki

didn't return to a pre-symbolic algebra era, but he did manage a conversational flow that kept the symbols to a bare and understandable minimum. His writing was a pleasure to read.

Well, this essay is not particularly significant, but I felt I had to respond to such a remarkable coincidence of seeing two books praised that held a special place in my mathematical life as well. (I also hope Bourbaki is dead and buried.)

References

[1] Yau, Shing-Tung and Steve Nadis, *The Shape of a Life, One Mathematician's Search for the Universe's Hidden Geometry*, Yale University Press, 328 pages, 2019

[2] Gamow, George, *One Two Three … Infinity: Facts and Speculations of Science*, Dover Publications Ind., New York, 1947, rev 1961, 1988 (https://archive.org/details/10.GeorgeGamowOneTwoThree...Infinty19471988)

[3] Simmons, George F., *Introduction to Topology and Modern Analysis*, 1963, Krieger Publishing Company; Reprint edition (1983) (http://susanka.org/HSforQM/[Simmons]_Introduction_to_Topology_and_Modern_Analysis.pdf)

[4] Kelley, John L., *General Topology*, D. Van Nostrand Company Inc. 1955 (https://archive.org/download/GeneralTopology/Kelley-GeneralTopology.pdf)