Consecutive Product Square

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This problem from Colin Hughes at *Maths Challenge*[1] is a most surprising result that takes a bit of tinkering to solve.

Problem

We can see that $3 \times 4 \times 5 \times 6 = 360 = 19^2 - 1$. Prove that the product of four consecutive integers is always one less than a perfect square.

The result is so mysterious at first that you begin to understand why the ancient Pythagoreans had a mystical relationship with mathematics.

Solution

Rather than just write down the answer, I thought it might be interesting to show how I figured it out. Basically, it was by considering several examples and seeing patterns.

n(n–1)(n–2)(n–3)	$\mathbf{m^2 - 1}$	$\mathbf{m_n - m_{n-1}}$	**m**	**m**
$4\cdot3\cdot2\cdot1 = 24$	$= 5^2 - 1$		$2\cdot2 + 1 = 2\cdot2 + 1\cdot2 - 1$	$2(3\cdot2/2) - 1$
$5\cdot4\cdot3\cdot2 = 120$	$= 11^2 - 1$	$6 = 3\cdot2$	$3\cdot2 + 2\cdot2 + 1\cdot2 - 1$	$2(4\cdot3/2) - 1$
$6\cdot5\cdot4\cdot3 = 360$	$= 19^2 - 1$	$8 = 4\cdot2$	$4\cdot2 + 3\cdot2 + 2\cdot2 + 1\cdot2 - 1$	$2(5\cdot4/2) - 1$
$7\cdot6\cdot5\cdot4 = 840$	$= 29^2 - 1$	$10 = 5\cdot2$	$5\cdot2 + 4\cdot2 + 3\cdot2 + 2\cdot2 + 1\cdot2 - 1$	$2(6\cdot5/2) - 1$
$8\cdot7\cdot6\cdot5 = 1680$	$= 41^2 - 1$	$12 = 6\cdot2$	$6\cdot2 + 5\cdot2 + 4\cdot2 + 3\cdot2 + 2\cdot2 + 1\cdot2 - 1$	$2(7\cdot6/2) - 1$

From this I conjectured that the pattern was

$$n(n-1)(n-2)(n-3) = [(n-1)(n-2) - 1]^2 - 1$$

So I expanded the RHS to see if it equaled the LHS.

$$\begin{aligned}[(n-1)(n-2) - 1]^2 - 1 &= ([(n-1)(n-2) - 1] - 1)\,([(n-1)(n-2) - 1] + 1)\\ &= [n^2 - 3n + 2 - 2][(n-1)(n-2)]\\ &= n(n-3)(n-1)(n-2)\end{aligned}$$

Confirmed.

Maths Challenge Solution

Let

$$Q = n(n + 1)(n + 2)(n + 3) + 1 = n^4 + 6n^3 + 11n^2 + 6n + 1.$$

If Q is square, it must be of the form, $(n^2 + an + 1)^2$.

Expanding

[1] "Consecutive Product Square" Problem ID: 185 (01 Nov 2004) Difficulty: 3 Star at mathschallenge.net. "A three-star problem: a good knowledge of school mathematics and/or some aspects of proof will be required." (https://mathschallenge.net/problems/pdfs/mathschallenge_3_star.pdf)

$$(n^2 + an + 1)^2 = n^4 + 2an^3 + (a^2+2)n^2 + 2an + 1,$$

and comparing coefficients, $2a = 6 \Rightarrow a = 3$; checking: $a^2+2 = 11$.

Therefore, $$Q = (n^2 + 3n + 1)^2.$$

Hence, the product of four consecutive integers is always one less than a perfect square.