Marching Cadets and Dog Problem

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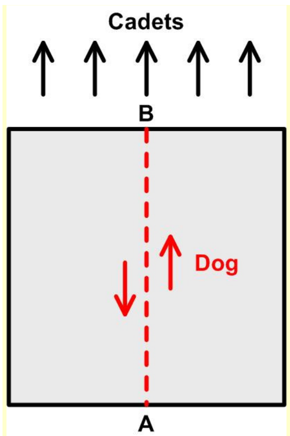
In my search for new problems I came across this one from Martin Gardner ([1] p.36).



A square formation of Army cadets, 50 feet on the side, is marching forward at a constant pace [see Figure]. The company mascot, a small terrier, starts at the center of the rear rank [position A in the illustration], trots forward in a straight line to the center of the front rank [position B], then trots back again in a straight line to the center of the rear. At the instant he returns to position A, the cadets have advanced exactly 50 feet. Assuming that the dog trots at a constant speed and loses no time in turning, how many feet does he travel?

Gardner gives a follow-up problem that is virtually impossible.

If you solve this problem, which calls for no more than a knowledge of elementary algebra, you may wish to tackle a much more difficult version proposed by the famous puzzlist Sam Loyd ([2] p.103). Instead of moving forward and back through the marching cadets, the mascot trots with constant speed around the *outside* of the square, keeping as close as possible to the square at all times. (For the problem we assume that he trots along the perimeter of the square.) As before, the formation has marched 50 feet by the time the dog returns to point A. How long is the dog's path?

My Solution to the First Problem

Figure 1 shows the dog starting at the back of the cadet ranks, running until it reaches the front of the ranks, and then reversing until it reaches the back again. The speeds noted on the figure correspond to the slopes of the lines. Since the dog runs at constant speed, its slope is constant (up to a reflection) and so makes a constant angle with respect to the horizontal. The section of its path beyond the 50 feet of the cadets forms an isosceles triangle, which implies that the distances forward and back D are the same and the times to travel them T_2 are the same.

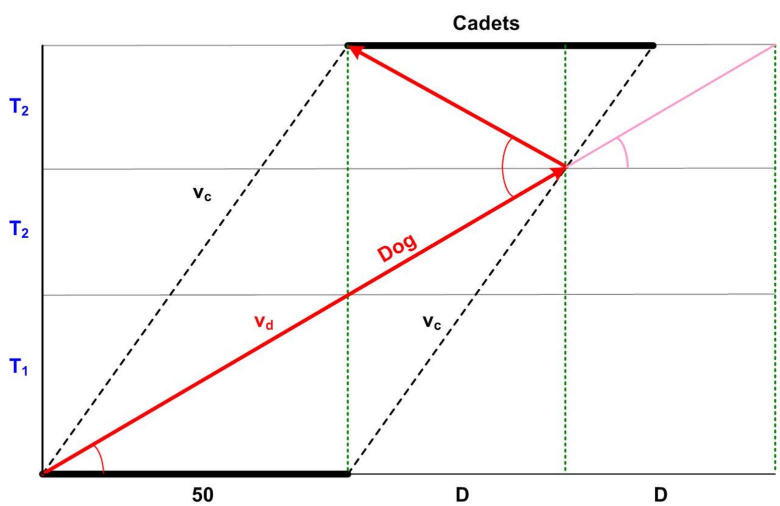


Figure 1 Space-time Diagram for Solution to First Problem

So from the geometry of Figure 1 we have the following equations.

$$v_c(T_1 + 2T_2) = 50 \tag{1}$$

$$v_d T_1 = 50 \tag{2}$$

$$v_d T_2 = D \tag{3}$$

$$50 + v_c(T_1 + T_2) = v_d(T_1 + T_2) \tag{4}$$

Equations (2) and (4) yield

$$50 + v_c(T_1 + T_2) = 50 + v_d T_2 \;\Rightarrow\; v_c(T_1 + T_2) = v_d T_2 \tag{5}$$

Equations (1) and (2) imply $$v_c(T_1 + 2T_2) = v_d T_1 \tag{6}$$

Eliminating v_c and then v_d from equations (5) and (6) results in

$$T_2 / (T_1 + 2T_2) = T_1 / (T_1 + T_2)$$

which, after cross multiplying, implies $$T_2 / T_1 = 1 / \sqrt{2} \tag{7}$$

Now $v_d T_1 = 50$ and $v_d T_2 = D$ implies $D / 50 = T_2 / T_1 = 1 / \sqrt{2}$

or $$D = 50 / \sqrt{2}$$

Since the total path traveled by the dog is 50 + 2D, we have that the dog traveled

$$50\,(1 + \sqrt{2}) \text{ feet} \approx 120.7 \text{ feet}$$

Martin Gardner's Solution to the First Problem

Let 1 be the length of the square of cadets and also the time it takes them to march this length. Their speed will also be 1. Let x be the total distance traveled by the dog and also its speed. On the dog's forward trip his speed relative to the cadets will be $x - 1$. On the return trip his speed relative to the cadets will be $x + 1$. Each trip is a distance of 1 (relative to the cadets), and the two trips are completed in unit time, so the following equation can be written:

$$\frac{1}{x-1} + \frac{1}{x+1} = 1 \tag{8}$$

This can be expressed as the quadratic: $x^2 - 2x - 1 = 0$, for which x has the positive value of $1 + \sqrt{2}$. Multiply this by 50 to get the final answer: 120.7+ feet. In other words, the dog travels a total distance equal to the length of the square of cadets plus that same length times the square root of 2.

My Solution to the Second Problem

I applied the same space-time diagram idea to the second problem (Figure 2), and in a similar fashion obtained the following equations.

Let T be the total time for the circumnavigation, D the total distance the dog ran, and v_c and v_d the speeds of the cadets and dog respectively. Then from the figure we have

$$T = 4T_1 + T_2 + T_3 \tag{9}$$

$$D = 4D_1 + 50 + D_2 + D_3 \tag{10}$$

and

$$50 = v_c\, T \tag{11}$$

$$D = v_d\, T \tag{12}$$

Now

$$D_1^2 = (v_d\, T_1)^2 = 25^2 + (v_c\, T_1)^2$$

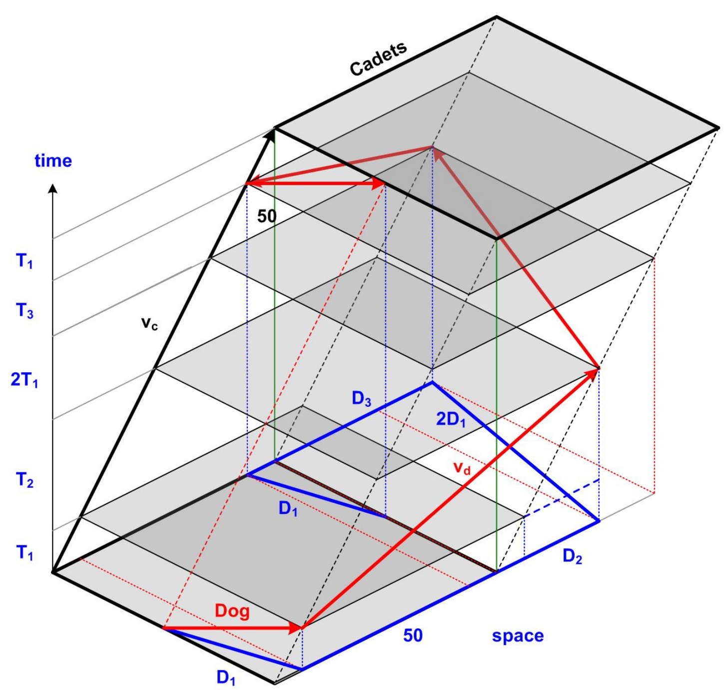


Figure 2 Space-time Diagram for Solution to Second Problem

implies $$T_1 = \frac{25}{\sqrt{v_d{}^2 - v_c{}^2}} \tag{13}$$

and $$v_d\, T_2 = 50 + v_c\, T_2$$

implies $$T_2 = \frac{50}{v_d - v_c} \tag{14}$$

and $$v_d\, T_3 = 50 - v_c\, T_3$$

implies $$T_3 = \frac{50}{v_d + v_c} \tag{15}$$

Combining equations (9) and (11) with (13), (14), and (15) yields

$$50 = v_c\left(2\frac{50}{\sqrt{v_d{}^2 - v_c{}^2}} + \frac{50}{v_d - v_c} + \frac{50}{v_d + v_c}\right) \tag{16}$$

Let $x = v_d / v_c$. Then from equations (11) and (12) we have $x = D / 50$ or $D = 50x$. So we can find the distance D the dog traveled if we can solve for x in the modified equation (16) given by

$$50 = 50\left(\frac{2}{\sqrt{x^2-1}} + \frac{1}{x-1} + \frac{1}{x+1}\right)$$

or $$1 = \frac{1}{x-1} + \frac{2}{\sqrt{x^2-1}} + \frac{1}{x+1} \tag{17}$$

Unfortunately, equation (17) has the dreaded radical. The usual practice is to put all the non-radicals on one side of the equation and the radical on the other. So equation (17) becomes

$$x^2 - 2x - 1 = 2\sqrt{x^2-1}$$

Squaring both sides and combining the results yields the hideous quartic equation

$$x^4 - 4x^3 - 2x^2 + 4x + 5 = 0 \tag{18}$$

At this point I figured I was way off base in my solution. After checking and rechecking, I could not find my mistake, so I gave up and looked at Martin Gardner's answer. Oh my.

Martin Gardner's Solution to the Second Problem

Loyd's version of the problem, in which the dog trots around the moving square, can be approached in exactly the same way. I paraphrase a clear, brief solution sent by Robert F. Jackson of the Computing Center at the University of Delaware.

As before, let 1 be the side of the square and also the time it takes the cadets to go 50 feet. Their speed will then also be 1. Let x be the distance traveled by the dog and also his speed. The dog's speed with respect to the speed of the square will be $x - 1$ on his forward trip, $\sqrt{x^2-1}$ each of his two transverse trips, and $x + 1$ on his backward trip. The circuit is completed in unit time, so we can write this equation:

$$\frac{1}{x-1} + \frac{2}{\sqrt{x^2-1}} + \frac{1}{x+1} = 1 \tag{19}$$

This can be expressed as the quartic equation: $x^4 - 4x^3 - 2x^2 + 4x + 5 = 0$. Only one positive real root is not extraneous: 4.18112+. We multiply this by 50 to get the desired answer: 209.056+ feet.[1]

Theodore W. Gibson, of the University of Virginia, found that the first form of the above equation can be written as follows, simply by taking the square root of each side:[2]

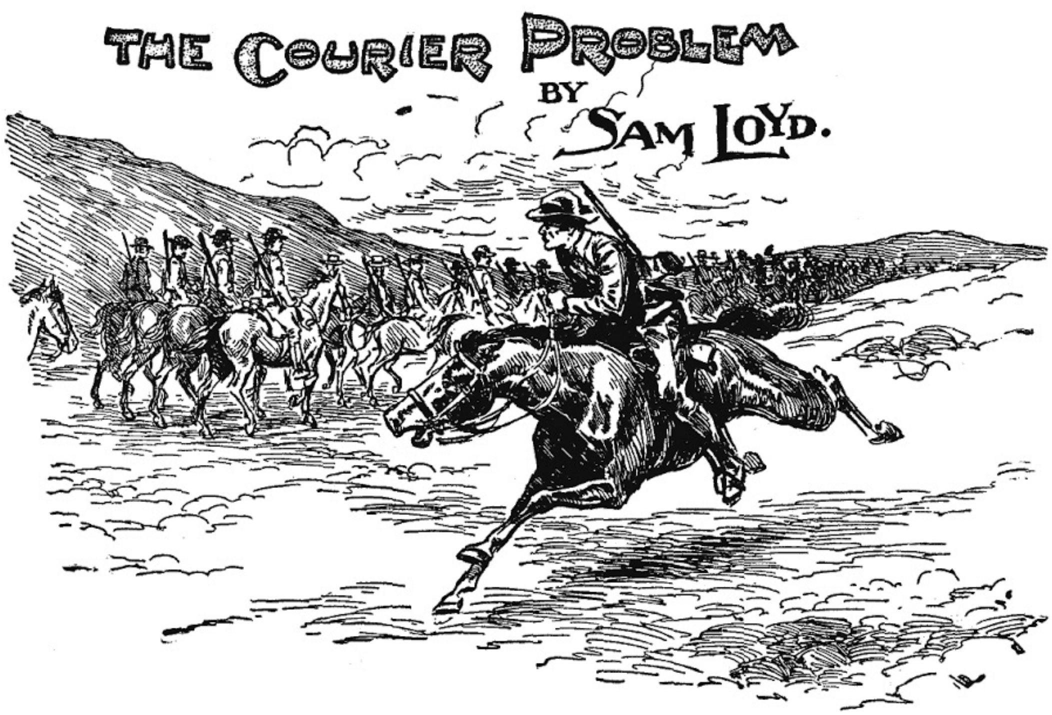
$$\frac{1}{\sqrt{x-1}} + \frac{1}{\sqrt{x+1}} = 1 \tag{20}$$

which is remarkably similar to the equation for the first version of the problem.[3]

Commentary

I must say the outcome for this second problem was most unexpected. I don't consider it fair to require a computer-generated solution, though it is possible one can use the closed formula (see https://en.wikipedia.org/wiki/Quartic_function), but that seemed too complicated and agonizing to contemplate. However, that raises the question of how did Sam Loyd solve the problem in the 19th century? I then tried to find out.

Martin Gardner's book was unavailable online, but I did find the problem through a 2008 article by Owen O'Shea ([3]). It turns out Loyd's formulation did not involve ranks of cadets and a dog, but rather an army and a courier. It was called "The Courier Problem" and could be found in Loyd's *Cyclopedia of Puzzles* published posthumously in 1914 ([4] p.315). Here is Loyd's statement:



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[1] JOS: Wait a minute. Where did this come from? Robert F. Jackson being of the Computing Center at the University of Delaware suggests he used a computer to get a numerical solution. I consider that to be cheating in a problem of this nature. Gardner's phrase "I paraphrase a clear, brief solution" must be his idea of a joke. He certainly "paraphrased" the solution into a brief statement. Anyway, at least I got the "right" answer, as far as I could get.

[2] JOS: Shades of the problem I posted in "Radical Radicals". This is cute, but we still have an equation with radicals (now two instead of one) that we cannot solve easily.

[3] JOS: The "similarity" is definitely in the beholder with those vicious radicals.

For the reason that many communications are being received relating to a very ancient problem, the authorship of which has been incorrectly accredited to me, occasion is taken to present the original version which has led to considerable discussion. It has been reproduced, in many forms, generally accompanied by an absurd statement regarding the impossibility of solving it, which produced letters of inquiry as well as correct answers from some, who, under the misapprehension of having mastered a hitherto unsolved problem, desire to have the same published.

It is a simple and pretty problem which yields readily to ordinary methods, and can be solved by experimental analysis upon the plan generally adopted by puzzlists. The trouble is that the terms of the problem are seldom given correctly and are not generally understood, for which reason, with the aid of a realistic picture, we will first look at the ancient version which appears in the oldest mathematical works:

> A courier starting from the rear of a moving army, fifty miles long, dashes forward and delivers a dispatch to the front and returns to his position in the rear, during the exact time it required the entire army to advance just fifty miles. How far did the courier have to travel in delivering the dispatch, and returning to his previous position in the rear of the army?

If the army were stationary he would clearly have to travel fifty miles forward and the same distance back. But under the circumstances as stated, he must go more than fifty miles to the front, as the army is steadily advancing; on his return trip he meets the army and therefore does not have to travel so far. To those who are familiar with the rules which govern the question it is a simple matter, but to most people it will prove to be a problem which can not be guessed off hand.

A better puzzle is created by the following extension of the theme given as problem No. 2:

> If a square army, fifty miles long by fifty miles wide, advances fifty miles while a courier makes the complete circuit of the army and returns to the starting point in the rear, how far does the courier have to travel?

Okay, so how did Loyd solve the problem?

Following the rule for solving puzzles of this kind, which is to multiply the length of the army by its length; then divide by 2 and the square root multiplied by 2 and added to the length of army will give the answer, we find that the courier travels a little over 120 miles.

In the second proposition the courier would have to travel a little over 209 miles.

What? That’s it? What rule? Where did the “rule” come from? Loyd’s idea of a mathematical explanation is certainly deficient by today’s standards. Furthermore, it is odd to see in modern times a verbal description of mathematics that does not use equations. It was like decoding a Medieval Arabic problem. If I understand the words, Loyd said the answer to the first problem, if L is the length of the army, is

$$2\sqrt{\frac{L^2}{2}} + L$$

This seems a strange way to write $\sqrt{2}\,L + L$ or $L(1 + \sqrt{2})$. But it is his “solution” to the second problem that is most startling. I guess in the past people were accustomed to providing answers to problems rather than solutions.

Martin Gardner Approach. One additional remark. Martin Gardner’s approach of casting the problem into units where the speed of the cadets was 1, the total time 1, and the speed of the dog *x* was essentially equivalent to my considering the ratio $x = v_d / v_c$ and having the Ts cancel.

Sam Loyd Rant. Okay, one more comment. In my search for new problems I went through a number of those by Sam Loyd. Invariably they seemed to include tacit assumptions that were not immediately evident from the statement of the problem. Most problem-givers make explicit if they are considering some factors needed to solve the problem. Otherwise, in the interests of keeping a problem simple, unmentioned constraints can normally be ignored. So there is a whiff of arbitrariness about Loyd's problems that I don't appreciate. You end up thinking of all sorts of extra conditions he did not mention and wonder if they are relevant. Some are and some aren't. In other words, you are trying to discover what the problem really is before you try to solve it—and there may be several legitimate possibilities to choose from. Not good. Also, as the current problem illustrates, his computations can get pretty ugly, which was perhaps normal for the 19th century but a bit inelegant by modern standards.

Numerical Solution of Quartic Equation. This is really the last comment. I couldn't resist trying to solve the quartic $f(x) = x^4 - 4x^3 - 2x^2 + 4x + 5 = 0$ numerically. I programmed Igor using Newton's iterative method for finding the zero of a differentiable function (Figure 3).

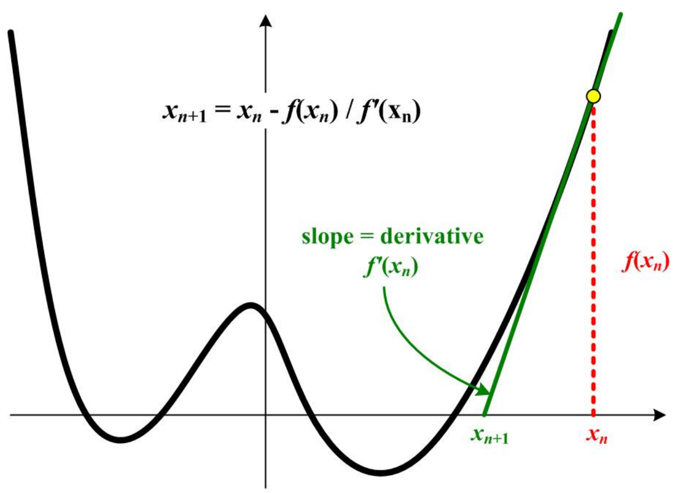

Figure 3 Newton's Method of Finding a Zero of Function $f(x)$

The original problem involved a quadratic (two solutions), but squaring introduced two new irrelevant solutions. So I started the search with a large positive value for x, hoping to arrive at the most positive and real solution. Actually starting with $x = 5$ proved sufficient. I got convergence after 5 iterations to seven decimal places with the result $x = 4.1811254$, which agrees with Martin Gardner's result from Robert F. Jackson. (Starting at $x = 1$, I got convergence in 4 iterations to $x = 1.3673385$. This value is less than $1 + \sqrt{2}$, which is the solution to the first problem entailing a shorter path, so it can be dismissed. Other starting points either didn't converge or resulted in the same two solutions.)

References

[1] Gardner, Martin, "Eight Problems," in *New Mathematical Diversions*, revised edition, Mathematical Association of America, 1995, original edition, 1966. Original article, February 1960.

[2] Gardner, Martin, ed., *More Mathematical Puzzles of Sam Loyd*, Vol. 2, Dover Publications, New York, 1960.

[3] O'Shea, Owen, "Sam Loyd's Courier Problem with Diophantus, Pythagoras, and Martin Gardner" *The College Mathematics Journal*, Vol. 39, No.5, November 2008. pp.387-391

[4] Loyd, Sam, *Cyclopedia of Puzzles*, Lamb Publishing, New York, 1914 (http://djm.cc/library/Cyclopedia_of_Puzzles_Loyd.pdf)