Lucky 7 Problem

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This interesting problem comes from Colin Hughes at the Maths Challenge website.[1]

Problem

Prove that for any number that is not a multiple of seven, then its cube will be one more or one less than a multiple of 7.

Solution

Given n is any number that is not evenly divisible by 7, let $n = 7q + r$, where the remainder $r = 1, 2, 3, 4, 5, 6$ and the multiple $q = 0, 1, 2, \ldots$. Then

$$n^3 = (7q + r)\,(7q + r)\,(7q + r) = 7 \times (\text{sum of various integral products}) + r^3 = 7\,m + r^3$$

for some integer m. Now consider r^3 for $r = 1, 2, 3, 4, 5, 6$

$$1^3 = 1 = 7\cdot 0 + 1$$

$$2^3 = 8 = 7\cdot 1 + 1$$

$$3^3 = 27 = 7\cdot 4 - 1$$

$$4^3 = 64 = 7\cdot 9 + 1$$

$$5^3 = 125 = 7\cdot 18 - 1$$

$$6^3 = 216 = 7\cdot 31 - 1$$

Therefore $r^3 = 7\,s \pm 1$ for some integer s

and so $n^3 = 7\,k \pm 1$ for some integer k

In terms of modulo arithmetic, $n^3 \equiv \pm 1 \bmod 7$, where n is not a multiple of 7.

Corollary

$n^3 \equiv -1, 0, 1 \bmod 7$, where $n \in \mathbf{N}$.

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[1] "Cubes And Multiples Of 7" Problem ID: 25 (Nov 2000) Difficulty: 3 Star, at MathsChallenge.net. "A three-star problem: a good knowledge of school mathematics and/or some aspects of proof will be required."