A Tricky Product

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This is a tricky product problem from Alfred Posamentier ([1] p.111) which naturally has a slick solution—if you can think of it.

Find the numerical value of the following expression:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\left(1-\frac{1}{25}\right)\cdots\left(1-\frac{1}{225}\right)$$

Solution

The trick is not to immediately try to multiply it out, but realize the pattern of products of differences of squares suggests replacing each term with its factorization, that is, notice that

$$(1 - x^2) = (1 - x)\ (1 + x)$$

Then the product becomes

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\left(1+\frac{1}{5}\right)\cdots\left(1-\frac{1}{15}\right)\left(1+\frac{1}{15}\right)$$

Now putting the terms over common denominators yields

$$\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)\left(\frac{4}{5}\right)\left(\frac{6}{5}\right)\cdots\left(\frac{13}{14}\right)\left(\frac{15}{14}\right)\left(\frac{14}{15}\right)\left(\frac{16}{15}\right)$$

or grouping terms more explicitly

$$\left(\frac{1}{2}\right)\ \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)\ \left(\frac{4}{3}\right)\left(\frac{3}{4}\right)\ \left(\frac{5}{4}\right)\left(\frac{4}{5}\right)\ \left(\frac{6}{5}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{14}{13}\right)\left(\frac{13}{14}\right)\ \left(\frac{15}{14}\right)\left(\frac{14}{15}\right)\ \left(\frac{16}{15}\right)$$

which yields

$$\left(\frac{1}{2}\right)\left(\frac{16}{15}\right)=\frac{8}{15}$$

References

[1] Posamentier, Alfred S., *Math Charmers: Tantalizing Tidbits for the Mind*, Prometheus Books, New York, 2003