Vector Sum Problem

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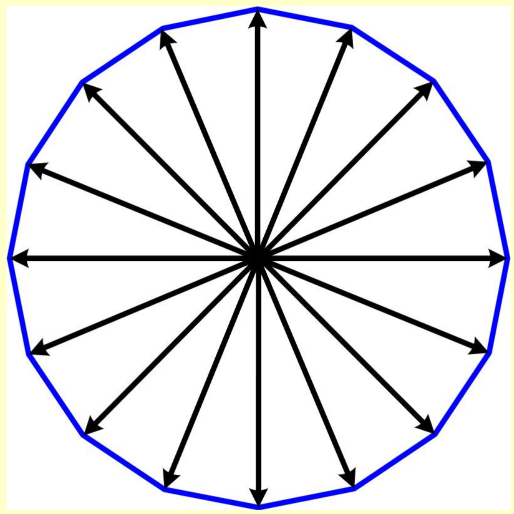
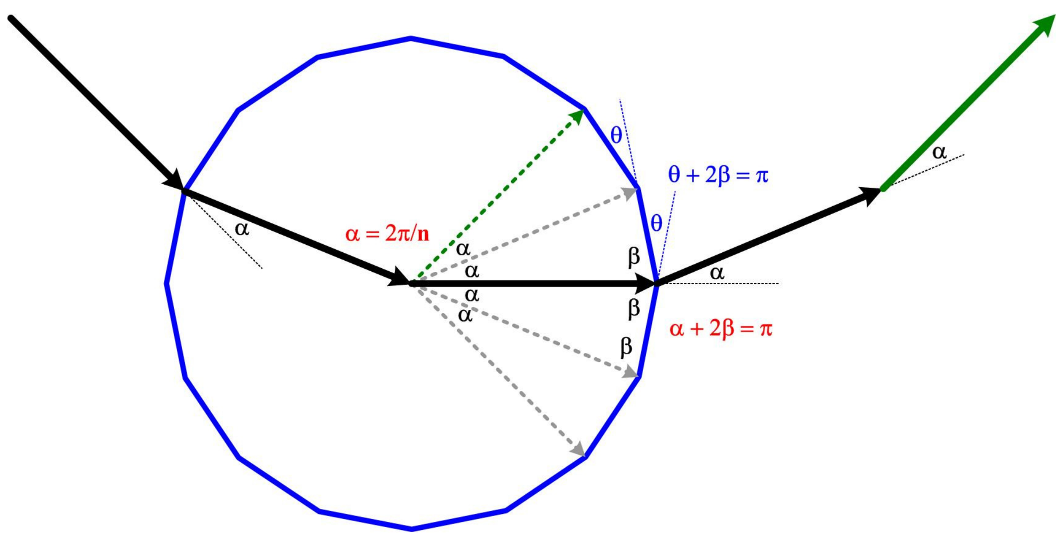
This is a fun problem from *Mathematical Quickies* (1967) ([1] p.58).

213. Prove that the sum of the vectors from the center of a regular polygon of n sides to its vertices is zero.



My Solution

The following figure practically accomplishes the proof without words. We take the vectors from



the center and parallel translate them to add head to tail in the usual vector addition. We see that a second, larger image of the n-polygon is generated by the summed vectors since they make the same angle with respect to each other as the sides of the original polygon, namely, α $(= 2\pi/n)$. This follows from the key relationship, namely, if α is the angle at the vertex of the isosceles triangle formed by adjacent vectors from the center and β is one of the base angles of the triangle, then $\alpha + 2\beta = \pi$. But as the figure shows, the angle θ that successive sides make with each other also satisfies $\theta + 2\beta = \pi$. So $\theta = \alpha$. Since the polygon made from the summed vectors is closed, the vector sum is zero (it returns to the center of the n-polygon).

Mathematical Quickies offers a different solution, but I find mine more explicitly evident.

Mathematical Quickies Solution

Let R be the resultant of the vectors. About the center, rotate the configuration through $2\pi/n$ radians, bringing it into coincidence with the former position. Note that R also rotates through $2\pi/n$ radians to become R'. Clearly, $R = R'$, but since their directions differ, then $R = R' = 0$.[1]

References

[1] Trigg, Charles W., *Mathematical Quickies: 270 Stimulating Problems with Solutions*, McGraw-Hill Publ, New York, 1967, corrected ed, Dover Publ., Mineola, New York, 1985.

[1] JOS: I am afraid I find this last statement a bit of handwaving that is not quite convincing. What does it mean to say "$R = R'$, but since their directions differ"? If two vectors are equal, their directions *do* equal. Through symmetry there appears to be a cogent argument, but I don't quite see it at the moment.