Six Squares Problem

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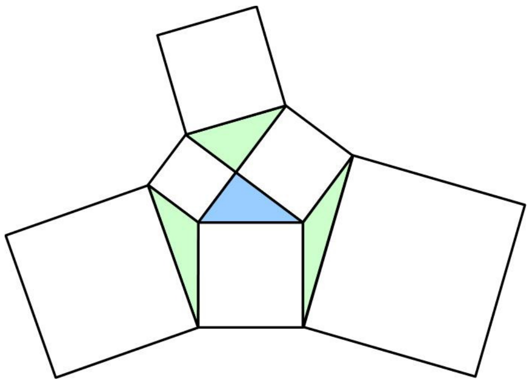
Jim Stevenson



This is a problem from the UKMT Senior Challenge for 2001. (It has been slightly edited to reflect the colors I added to the diagram.)

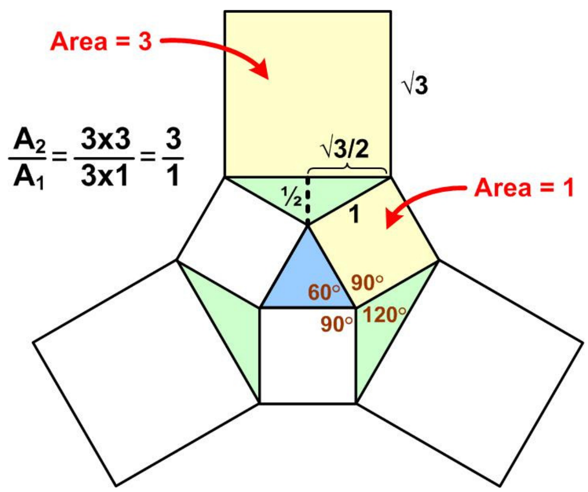
The [arbitrary] blue triangle is drawn, and a square is drawn on each of its edges. The three green triangles are then formed by drawing their lines which join vertices of the squares and a square is now drawn on each of these three lines. The total area of the original three squares is A_1, and the total area of the three new squares is A_2. Given that $A_2 = k A_1$, then

A k = 1 B k = 3/2 C k = 2 D k = 3
E more information is needed.

I solved this problem using a Polya principle to simplify the situation, but UKMT's solution was direct (and more complicated).

My Solution

Assuming the problem is honest, that is, there are no hidden assumptions, then we conclude the result holds for all triangles, that is, k is a constant independent of the blue triangle. So choose a "nice" triangle to consider, namely, an equilateral triangle (See the figure at right).



Then the interior angles of the blue triangle are all 60° and the obtuse angle of the green triangle is 360° – (2 x 90° + 60°) = 120° But since it is isosceles, the perpendicular from the obtuse angled vertex to the base of the green triangle also bisects the angle, making two 60-30 right triangles. If we assume the small squares all have edge 1 (and thus area 1), then the edge of each of the large squares is 2 x (√3 / 2) = √3 (and thus have area 3). Therefore,

$$A_2 / A_1 = (3 \times 3) / (3 \times 1) = 3$$

which implies k = 3 (Answer D).

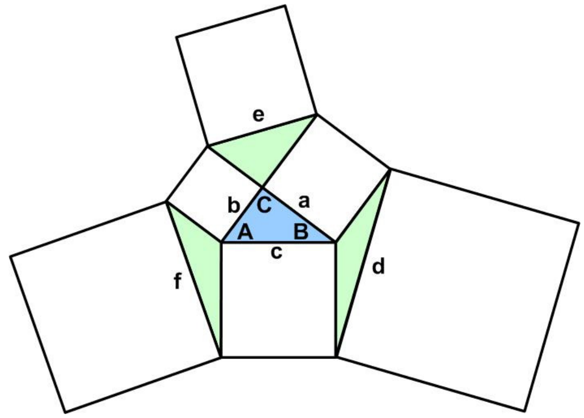
UKMT Solution

The UKMT solution effectively proves the result holds for any triangle by proving it directly for an arbitrary (blue) triangle.[1] But it resorts to the Law of Cosines to prove it. Labeling the vertices

[1] JOS: Perhaps my interpretation of the problem is idiosyncratic. I assumed that if the problem did not put additional constraints on the triangle, then you are allowed to choose any triangle you want. I recall this

(and corresponding interior angles) of the blue triangle as A, B, C and the sides opposite as a, b, c, respectively, as shown in the next figure, then the area of all the small squares is

$$A_1 = a^2 + b^2 + c^2.$$



Multiple applications of the Law of Cosines yields

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Similarly for the large squares, where the angle of the green triangle opposite A, for example, is, as we argued above,

$$360° - (2 \times 90° + A) = 180° - A.$$

Thus we have

$$f^2 = b^2 + c^2 - 2bc \cos (180° - A) = b^2 + c^2 + 2bc \cos A = b^2 + c^2 + ((b^2 + c^2) - a^2) = 2(b^2 + c^2) - a^2$$

$$d^2 = a^2 + c^2 - 2ac \cos (180° - B) = a^2 + c^2 + 2ac \cos B = a^2 + c^2 + ((a^2 + c^2) - b^2) = 2(a^2 + c^2) - b^2$$

$$e^2 = a^2 + b^2 - 2ab \cos (180° - C) = a^2 + b^2 + 2ab \cos C = a^2 + b^2 + ((a^2 + b^2) - c^2) = 2(a^2 + b^2) - c^2$$

Therefore

$$A_2 = f^2 + d^2 + e^2 = 4(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2) = 3(a^2 + b^2 + c^2) = 3 A_1$$

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idea from George Polya's famous and invaluable book, *How to Solve It* (1945), which should be on every mathematician's library shelf. But perhaps the problem is implicitly requiring you to *prove* the result is true for an arbitrary triangle. However, the statement in the problem "Given that $A_2 = k A_1$" where k is assumed to be a constant tells me that you can assume this is true for all triangles and so you can chose the one that makes the solution easiest. That is always the case when you are trying to solve for a constant like this.