Hitting the Target

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Here is a problem from the UKMT Senior (17-18 year-old) Mathematics Challenge for 2012:

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23. Tom and Geri have a competition. Initially, each player has one attempt at hitting a target. If one player hits the target and the other does not then the successful player wins. If both players hit the target, or if both players miss the target, then each has another attempt, with the same rules applying. If the probability of Tom hitting the target is always 4/5 and the probability of Geri hitting the target is always 2/3, what is the probability that Tom wins the competition?

A 4/15 B 8/15 C 2/3 D 4/5 E 13/15

My Solution

Here are the results of any turn and the probabilities associated with the various outcomes.

Results

Tom \ Geri	Hits	**Misses**
Hits	HH	HM
Misses	MH	MM

Probabilities

Tom \ Geri	Hits 2/3	**Misses 1/3**
Hits 4/5	8/15	**4/15**
Misses 1/5	**2/15**	1/15

The competition is only over when someone wins. So I ignored the occasions when there was no winner. When the competition is over (when someone wins), Tom has 2:1 odds of being the winner.[1] Therefore the probability of his winning the competition is 2/3. For me the only reason to consider the probabilities of no win is if we are to estimate how many turns it will take for a win. Otherwise, it is irrelevant.

I had no confidence in my answer (and reasoning), since invariably I get probability questions wrong. But since I got this one right, I thought I might include it in a post. I have added the UKMT solutions just to show how differently one can reason about these problems.

I thought UKMT Solution 1 was quite complicated. I have always been unsure about assigning probabilities to cases, that is, to "or" events where you "sum" the probabilities. Usually you think you should just add the probabilities. But care must be taken. In fact events A or B occur if and only if it is false that both A and B don't occur. That is the probability that A does not occur is 1 – prob(A) and similarly the probability that B does not occur is 1 – prob(B). So the probability that both these cases do not occur is 1 – (1 – prob(A))(1 – prob(B)) = prob(A) + prob(B) – prob(A)prob(B). It is that last term that causes trouble in summing probabilities, which is why I always approach "or" events through the negatives of the probabilities.

[1] ([1]) "The **odds** are defined as the probability that the event will occur divided by the probability that the event will not occur. If the probability of an event occurring is Y, then the probability of the event not occurring is 1-Y. (Example: If the probability of an event is 0.80 (80%), then the probability that the event will not occur is 1-0.80 = 0.20, or 20%. … So, in this example, if the probability of the event occurring = 0.80, then the odds are 0.80 / (1-0.80) = 0.80/0.20 = 4 (i.e., 4 to 1).)"

UKMT Solution 2 took me a while to understand. Both these solutions took into account the cases where neither Tom nor Geri won. But I did not. So perhaps I did not understand the situation sufficiently, but I feel my solution makes sense. This insecurity is why I don't like probability problems, nor even the subject for that matter.

UKMT Solutions

Solution 1

Tom wins after one attempt each if he hits the target and Geri misses. The probability of this happening is 4/5 x 1/3 = 4/15. Similarly the probability that Geri wins after one attempt is 2/3 x 1/5 = 2/15. So the probability that both competitors will have at least one more attempt is 1 – 4/15 – 2/15 = 3/5. Therefore the probability that Tom wins after two attempts is 3/5 x 4/15. The probability that neither Tom nor Geri wins after two attempts is 3/5 x 3/5. So the probability that Tom wins after three attempts each is $(3/5)^2$ x 4/15 and, more generally, the probability that he wins after n attempts each is $(3/5)^{n-1}$ x 4/15. Therefore the probability that Tom wins is

$$\frac{4}{15} + \left(\frac{3}{5}\right) \times \frac{4}{15} + \left(\frac{3}{5}\right)^2 \times \frac{4}{15} + \left(\frac{3}{5}\right)^3 \times \frac{4}{15} + \ \ldots .$$

This is the sum to infinity of a geometric series with the first term 4/15 and common ratio 3/5. Its value is ${}^4/_{15}$ / (1 – ${}^3/_5$) = ${}^2/_3$.

Solution 2

Let the probability that Tom wins the competition be p. The probability that initially Tom hits and Geri misses is 4/5 x 1/3 = 4/15. The probability that initially they both hit is 4/5 x 2/3 = 8/15 and that they both miss is 1/5 x 1/3 = 1/15. So the probability that either they both hit or both miss is 8/15 + 1/15 = 9/15 = 3/5. If they both hit or both miss the competition is in the same position as it was initially. So Tom's probability of winning is then p. Therefore, p = 4/15 + (3/5)p. So (2/5)p = 4/15 and hence p = 2/3.

References

[1] *The Difference Between "Probability" and "Odds"* Boston University School of Public Health, October 27, 2017 (http://sphweb.bumc.bu.edu/otlt/MPH-Modules/BS/BS704_Confidence_Intervals/BS704_Confidence_Intervals10.html, retrieved 4/14/2019)