Challenging Sum

16 April 2019

Jim Stevenson

***a* + *b* + *c* + *d* = ?** Here is a problem from the UKMT Senior (17-18 year-old) Mathematics Challenge for 2009:

Four positive integers *a, b, c,* and *d* are such that

$$abcd + abc + bcd + cda + dab + ab + bc + cd + da + ac + bd + a + b + c + d = 2009. \qquad (*)$$

What is the value of $a + b + c + d$?

A 73 B 75 C 77 D 79 E 81

Solution

The key is to notice that equation (*) is a sum of elementary symmetric functions or polynomials in the four variables *a, b, c, d*. Why would you notice that? I was introduced to these in an abstract algebra course as a lead up to the general solution to polynomial equations of arbitrary degree via Galois theory—not the usual subject for seniors in high school I would have imagined, more like upper level college. But I guess things have advanced these days.

Historically, it was noticed that for polynomial equations of successively higher degrees there was a pattern linking the roots of the equations, say α, β, γ, δ, …, to the coefficients of the polynomials. For example, for quadratic polynomials

$$x^2 + a_1x + a_0 = (x - \alpha)(x - \beta)$$
$$= x^2 - (\alpha + \beta)x + \alpha\beta = x^2 - s_1(\alpha, \beta)x + s_2(\alpha, \beta)$$

where the following are the elementary symmetric polynomials in two variables

$$s_1(\alpha, \beta) = \alpha + \beta$$

$$s_2(\alpha, \beta) = \alpha\beta$$

and for polynomials of degree 3

$$x^3 + a_2x^2 + a_1x + a_0 = (x - \alpha)(x - \beta)(x - \gamma)$$
$$= x^3 - s_1(\alpha, \beta, \gamma)x^2 + s_2(\alpha, \beta, \gamma)x - s_3(\alpha, \beta, \gamma)$$

where the following are the elementary symmetric polynomials in three variables

$$s_1(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$$

$$s_2(\alpha, \beta, \gamma) = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$s_3(\alpha, \beta, \gamma) = \alpha\beta\gamma$$

and finally for polynomials of degree 4

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta) \qquad (**)$$
$$= x^4 - s_1(\alpha, \beta, \gamma, \delta)x^3 + s_2(\alpha, \beta, \gamma, \delta)x^2 - s_3(\alpha, \beta, \gamma, \delta)x + s_4(\alpha, \beta, \gamma, \delta)$$

where the following are the elementary symmetric polynomials in four variables

$s_1(\alpha, \beta, \gamma, \delta) = \alpha + \beta + \gamma + \delta$

$s_2(\alpha, \beta, \gamma, \delta) = \alpha\beta + \beta\gamma + \gamma\alpha + \alpha\delta + \beta\delta + \gamma\delta$

$s_3(\alpha, \beta, \gamma, \delta) = \alpha\beta\gamma + \alpha\beta\delta + \alpha\delta\gamma + \delta\beta\gamma$

$s_4(\alpha, \beta, \gamma, \delta) = \alpha\beta\gamma\delta$

Notice that swapping any pair of roots in the symmetric polynomials leaves their value unchanged, and thus the symmetry.

Now if

$$f(x) = x^4 + s_1(a, b, c, d)x^3 + s_2(a, b, c, d)x^2 + s_3(a, b, c, d)x + s_4(a, b, c, d)$$

(notice the lack of negative signs), then from equation (*), the definition of the elementary symmetric polynomials in four variables, and equation (**) with all negative roots, we have

$$f(1) = (1 + a)(1 + b)(1 + c)(1 + d) = 2009 + 1 = 2010$$

Since a, b, c, d are restricted to positive integers, we need to factor 2010 into a product of (hopefully) four unique integers:

$$2010 = 2 \cdot 3 \cdot 5 \cdot 67$$

Therefore, $a = 1$, $b = 2$, $c = 4$, and $d = 66$, so that $a + b + c + d = 73$, answer A.