Tandem Circles

31 March 2019

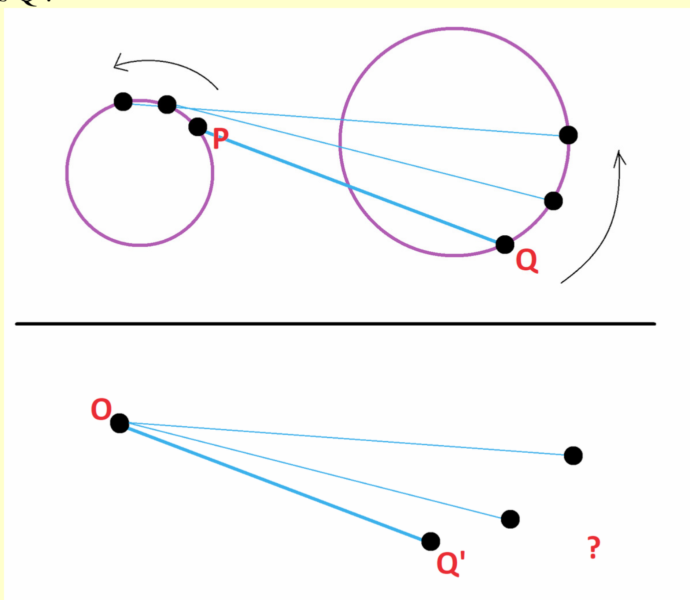
Jim Stevenson

James Tanton had another interesting puzzle on Twitter.

https://twitter.com/jamestanton/status/1111258545599602689

James Tanton, 28 March 2019

Points P and Q each move counterclockwise on a circle, uniform speed, one revolution per minute. At each instant, segment PQ is translated so that P is at the origin. Let Q' be the image of Q. What curve is traced by the points Q'?



Solution

There may be an easier direct geometric solution, but I thought the problem suggested complex variables. Figure 1 shows a representation of the problem in complex variables, with z corresponding to the point P on the first circle and z' corresponding to the point Q on the second circle.

Assuming that z' rotates around its circle at the same rate and direction as z, then its argument is also the same as for z, namely, θ. Furthermore, Tanton's figure seems to indicate a constant phase offset of z', represented by θ_0. Let z_0 be the complex variable representing the separation of the centers of the two circles and w be the complex variable representing the line segment

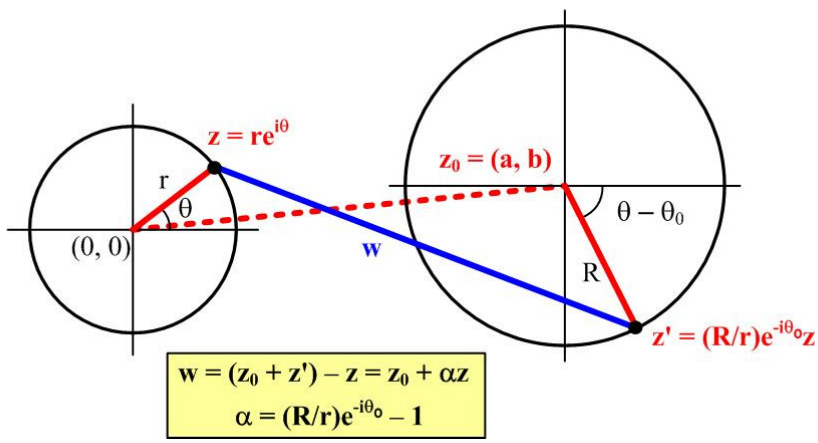


Figure 1 Complex Variable Representation of the Problem

PQ. Then we are interested in seeing what a plot of w looks like in the complex plane.

At first, I thought if might produce an ellipse, but after performing the calculations shown in Figure 1, I realized it sweeps out a circle around z_0 with radius equal to $|\alpha|r$, where α is a complex constant that collects all the differences between the two circles. The fact that α is constant means we get another circle for the plot of Q' = w, which is a rotated, shrunk or expanded version of the original second circle (Figure 2).

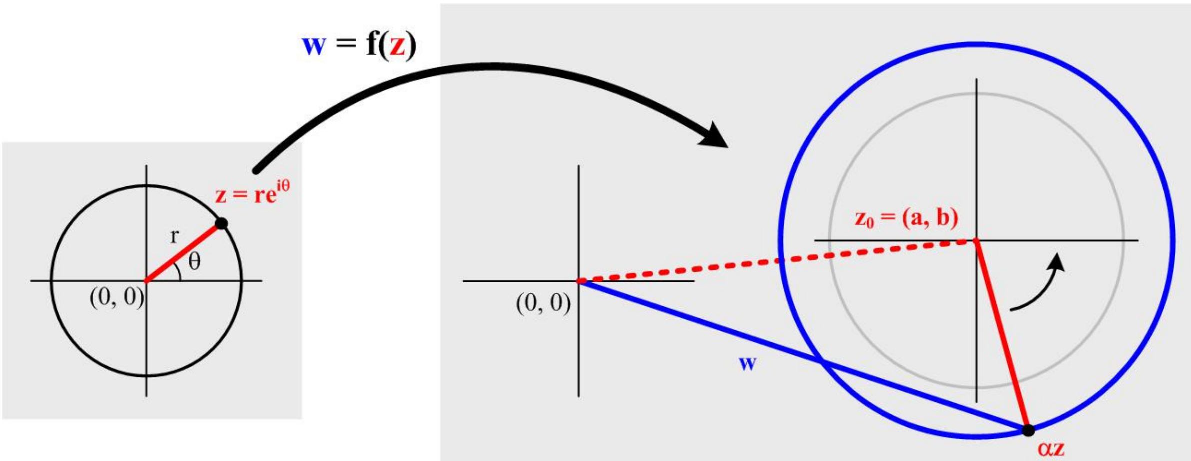


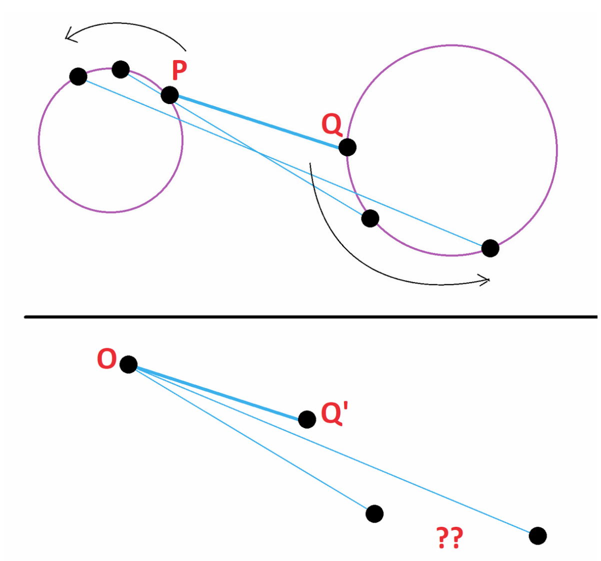
Figure 2   Plot of w (aka Q') in the complex plane.

Addendum

On 29 March 2019, James Tanton added the following:

(https://twitter.com/jamestanton/status/1111625430723751936)

And of course, a la @AlexKontorovich: P & Q each move on a circle uniform speed, one revolution per min, but in reverse directions. At each instant, segment PQ is translated so that P at the origin. Image of Q is Q'. What curve is traced by the Q'? (Re yesterday, again a circle?)



First, the diagram as shown is just the same as the previous diagram (the motion around both circles is counter-clockwise), but with a different phase offset. So the answer is the same.

If the intent was to have Q move around the second circle in a clockwise direction, then instead of αz in Figure 2 we have $\alpha\,\bar{z}$, the complex conjugate. The new circle has the same radius as before, but θ becomes $-\theta$, that is, the rotation of w is now clockwise.