River Crossing

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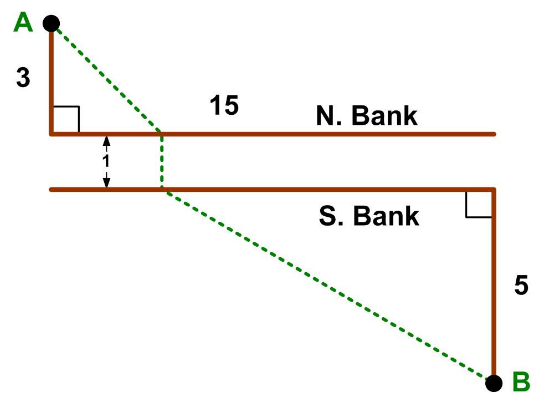
This is a riff on a classic problem given in *Challenging Problems in Algebra* ([1] p.49):

N. Bank and S. Bank are, respectively, the north and south banks of a river with a uniform width of one mile. Town A is 3 miles north of N. Bank, town B is 5 miles south of S. Bank and 15 miles east of A. If crossing at the river banks is only at right angles to the banks, find the length of the shortest path from A to B.



Challenge. If the rate of land travel is uniformly 8 mph, and the rowing rate on the river is $1^2/_3$ mph (in still water) with a west to east current of $1^1/_3$ mph, find the shortest time it takes to go from A to B. [The path across the river is must still be perpendicular to the banks.]

Solution

Since the distance across the river is one mile no matter where it occurs, and since the paths across land have to end at the same points across from each other, we can remove the river and collapse the figure to find the minimal distance over land (Figure 1). The minimal distance of course will be a straight line between A and B, which is the hypotenuse of the right triangle with legs 8 and 15. This turns out to be 17. So the overall minimal distance, including the irreducible 1 mile across the river, is 18 miles.

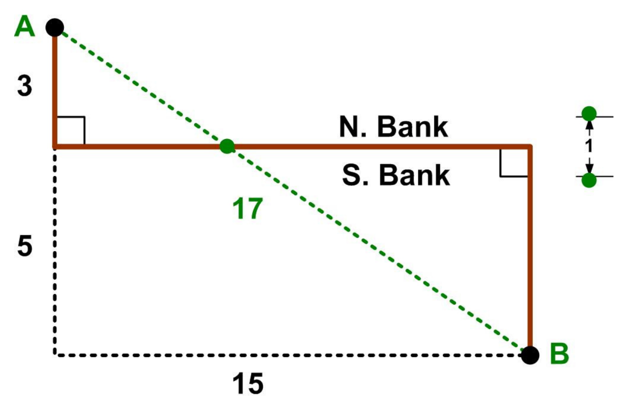


Figure 1 Shortest Distance

Challenge Solution.

The minimal time over land will be the minimal distance divided by the speed 8 mph, or $17/8 = 2^1/_8$ hours. So now we need to consider the time it takes to cross the river, which will be a constant.

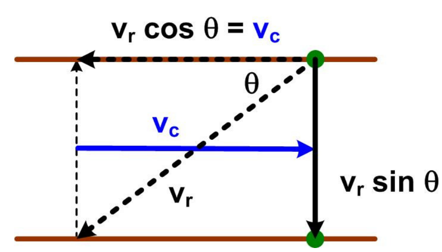


Figure 2 Time Crossing River

Let $v_r = 5/3$ mph be the speed of rowing and $v_c = 4/3$ be the speed of the river current. Then in order for the boat to cross perpendicularly to the banks it most overcome the speed of the current. Since velocities add vectorially, the direction the boat must be rowed (angle θ) must be such that the projection of the boat's speed horizontally exactly equals the speed of the current. Therefore $v_r \cos \theta = v_c$ or $(5/3) \cos \theta = (4/3)$ or $\cos \theta = 4/5$. Now the actual speed and direction of the boat will then be $v_r \sin \theta = (5/3)(3/5) = 1$ mph, perpendicularly across the river. Since the river is a mile wide, it will take 1 hour to cross. Therefore the total minimal time to travel from A to B is $1 + 2^1/_8 = 3^1/_8$ hours.

References

[1] Posamentier, Alfred S., and Charles T. Salkind, *Challenging Problems in Algebra*, MacMillan Co, 1970, slightly revised introduction, Dover Publications, Mineola, New York, 1996.