# The Essence of Mathematics

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https://www.sciencedaily.com/images/2014/11/141113152916_1_540x360.jpg Credit: © agsandrew / Fotolia

It is a bit presumptuous to think I could reduce the universe of mathematics to some succinct essence, but ever since I first saw a column in Martin Gardner's *Scientific American Mathematical Games* in 1967, I thought his example illustrated the essential feature of mathematics, or at least one of its principal attributes. And he posed it in a way that would be accessible to anyone. I especially wanted to credit Martin Gardner, since the idea resurfaced recently, uncredited, in some attractive videos by Katie Steckles and James Grime ([1]). (This reminds me of the Borges idea that "eighty years of oblivion are perhaps equal to novelty" ([2]).)

Here is the relevant excerpt from Martin Gardner's column ([3]):

- Nine playing cards, with values from ace to nine, are face up on the table. Players take turns picking a card. The first to obtain three cards that add to 15 is the winner. …
- Each of the following words is printed on a card: HOT, HEAR, TIED, FORM, WASP, BRIM, TANK, SHIP, WOES. The nine cards are placed face up on the table. Players take turns removing a card. The first to hold three cards that bear the same letter is the winner. (The Canadian mathematician Leo Moser, who devised this game, called it "Hot.")

For each game the question is: If both players make their best moves, is the game a win for the first player, a win for the second player, or a draw? Perhaps the reader has already experienced what the Gestalt psychologists call "closure" and recognized that [both] games are isomorphic with ticktacktoe!

It is easy to see that this is the case. For the first game we make a list of all the triplets of distinct digits from 1 to 9 that have a sum of 15. There are exactly eight such triplets. They can be interlocked on a ticktacktoe board as shown in Figure 88 to form the familiar order-3 magic square on which every row, column, and main diagonal is one of the triplets. Each numbered card drawn by a player corresponds to a ticktacktoe play on the cell of the magic square that bears that digit. Each set of triplets that wins in the card game corresponds to a winning ticktacktoe row on the magic square. Anyone who can play a perfect game of ticktacktoe and who also memorizes the magic square can immediately playa perfect game in this card version. …

| | | |
|---|---|---|
| 2 | 9 | 4 |
| 7 | 5 | 3 |
| 6 | 1 | 8 |

**FIGURE 88** *Ticktacktoe version of card game*

> The isomorphism of Moser's word game and ticktacktoe becomes obvious when the nine words are written inside the cells of a ticktacktoe matrix as shown in Figure 90. Each set of three-in-a-row words has a common letter, and there are no such sets other than the eight displayed in this way. Again, memorizing the square of words instantly enables a perfect-game ticktacktoe player to play a perfect game of Hot. Since ticktacktoe played rationally is always a draw, the same is true of the … equivalent games, although the first player naturally has a strong advantage over a second player who is not aware that he is playing disguised ticktacktoe or who may not play a perfect game of ticktacktoe.
>
> | | | |
> |---|---|---|
> | HOT | FORM | WOES |
> | TANK | HEAR | WASP |
> | TIED | BRIM | SHIP |
>
> **FIGURE 90** *Key to the game of Hot*
>
> One who grasps the essential identity of the three games will have obtained a valuable insight; mathematics abounds with "games" that seem to have little in common and yet are merely two different sets of symbols and rules for playing the *same* game. Geometry and algebra, for example, are two ways of playing exactly the same game, as Descartes's great discovery of analytic geometry shows. …

The idea is illustrated succinctly in Figure 1. The essence of the two games is abstracted in the form of "knowledge representation" (to use a 1980s artificial intelligence term) or a model that preserves the objects in the games and the actions between them, albeit in a different form. Namely, both games involve 9 objects, numbers in the first and words in the second. Both games involve being the first to choose 3 of the objects that satisfy a criterion, summing to 15 in the first game and having a unique common letter in the second game. Via the interim mapping both games assign their objects to spaces in the tic-tac-toe grid and both satisfy their winning criterion via the three-in-a-row win in tic-tac-toe. This sameness of objects and relationships is captured in the word "isomorphism."

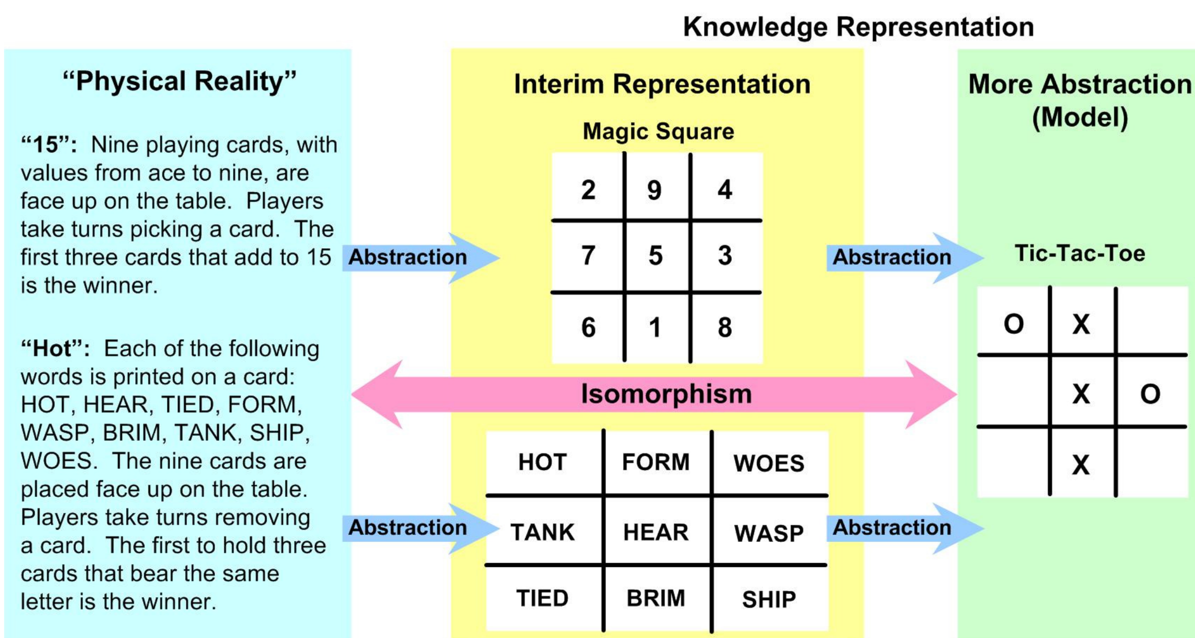


**Figure 1 Martin Gardner Example of Mathematical Knowledge Representation**

The other significant aspect to these games is that through an ingenious transformation they are mapped to a model whose behavior is completely understood. The strategy for playing the "15" card game and Hot is then immediately known from the knowledge of tic-tac-toe. The game of tic-tac-toe has been analyzed on its own for centuries. In fact, after coming to an understanding of all the strategies for the 3 x 3 game, natural questions as to what about a 4 x 4 game? or about an n x n game? What about 3-dimensional tic-tac-toe? 3 x 3 x 3? etc.

# Physical Reality to Mathematics Connections

So we have three elements in this view: the original "physical reality", the transformation or interim representation, and then the final abstraction or model. Mathematicians, especially so-called "pure" mathematicians, are primarily engaged with the third component of this triad. They thoroughly explore and question objects and relationships, possibly initially suggested by a connection with physical reality, but ultimately driven by the inner logic of the objects themselves and even their relationships with other objects under study. This activity often leads the mathematician to create new structures and systems that seem to have no obvious relationship to physical reality.

One might say "applied" mathematicians or physicists concentrate on the interim transformations between "physical reality" and the mathematical models. As the Gardner examples show, coming up with the interim transformations takes a lot of ingenuity and creativity. It often requires knowledge of existing models or the ability to sketch new models, which then have to be made rigorous by the mathematicians. In some cases, the sketchy new models of the physicists point the way to already existing structures the mathematicians have studied on their own. One example out of many would be Heisenberg's early formulation of quantum mechanics in 1925 that others realized was an application of matrices developed by the mathematician Cayley decades before. On the other hand, the sketchy ideas of the physicists can lead to new mathematics, such as the theory of distributions that grew out of the quantum physicists' idea of a Dirac delta function $\delta$ that was 0 everywhere except at the origin where it was "infinitely" large to enable its integral from $-\infty$ to $+\infty$ to be 1 (as stated this is nonsense; it took the mathematicians to create a logical theory (distributions) that captured the physicists' ideas).

The power of isomorphism is that it is an *exact* parallel between two regimes: there is a one-to-one correspondence between the objects in the two settings, and the actions or transformations between the objects in one setting are the same as those in the other setting—they have the same properties and characteristics. The two regimes are true mirrors of one another. But there are weaker parallels between situations captured in the idea of a mathematical model of some of the features in a setting. The mathematical structures and operations are not necessarily identical to the physical situation, but they capture some basic properties through the process of abstraction. How this is possible is where the magic and mystery of mathematics comes in—a mystery which is ably presented in Eugene Wigner's famous essay, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" ([4]). The great examples are Newton's laws of motion and gravitation that govern planetary motion, Einstein's Theory of General Relativity that supplanted Newton's scheme where motion under the force of gravity was replaced by geodesic paths in a four-dimensional curved space-time, and finally the Standard Model of Quantum Mechanics where conservation laws correspond to symmetries of Lie groups, thanks to the theorems of Emmy Noether.

I should add that this "mathematization" of physical reality is not limited to applied mathematicians and physicists. It spread to other physical sciences such as chemistry, geology, meteorology, and even biology, especially recently. It even has infected social sciences such as economics, which has been practically reduced to a branch of mathematics in the last 50 years. (Many question whether this is entirely a good thing.) Of course, probability and statistics have been a major tool in this invasion, but calculus and its associated differential equations have been a bulwark in mathematical modeling in these fields.

## Intra-Mathematics Connections

But within mathematics itself the representation of one mathematical system by another is one of the supreme joys of mathematical endeavors. These often hidden relationships are truly delightful when they are discovered—reflecting surprise and pleasure like that associated with a deft magic illusion.

As alluded to by Martin Gardner, perhaps the most famous connection occurred when Descartes (and Fermat) mapped plane geometry to symbolic algebra in the $17^{th}$ century. But even more subtly and profoundly was the actual creation of symbolic algebra itself by the end of the $16^{th}$ century after several thousand years of having to solve mathematical problems mostly rhetorically. That is, word problems could not be solved by mapping them to symbolic algebraic equations and manipulating them via arithmetic operations. They had to be unraveled mentally and with the possible help of computational tools such as an abacus. Other than plane geometry there were no mathematical models available to solve problems. It is true that plane geometry was used extensively for some 2000 years to solve what we now consider algebraic problems, such as finding solutions to quadratic and cubic equations. But these involved geometric constructions that were elaborate and difficult.

The arrival in the West of symbolic algebra by the $16^{th}$ century followed the introduction earlier from the East of the Indo-Arabic positional decimal number system. The modern trend is to down-play mastering the arithmetic procedures associated with this number system in favor of using electronic calculators. Certainly that makes sense for lengthy problems, but no student should be deprived of experiencing the marvel of a system of numerals (names for numbers) that contain their own method of computation. We learn how to add, subtract, multiply, and divide numbers by manipulating the digits and their positions in the numerals representing these numbers. How marvelous is that! What other names for objects contain such richness? That is a mathematical model par excellence, and it is available to anyone, not just a mathematician.

Grant Sanderson at his Youtube website 3blue1brown has some great examples of how mathematical models and representations can solve mathematically grounded problems, often in very imaginative ways. He explains how counting the number of collisions two moving masses create between themselves and a wall produces the decimal expansion for pi ([5]) and how dividing the spoils from a stolen necklace relates to the Borsuk-Ulam Theorem in topology ([6]). Many of the problems I have collected for my *Meditations on Mathematics* are solved with some of these surprising models, such as mod 3 arithmetic for the Barrel of Beer problem, the “skew billiard table” for the Three Jugs Problem, and Pascal’s Triangle in the Chalkdust Grid Problem. And of course most of the problems are solved the now classic way of mapping them to geometric figures (such as space-time diagrams), assigning algebraic relationships, and solving them algorithmically. I cringe whenever I hear a popularized book about mathematics proudly state how few equations it uses. That completely obliterates the essence of mathematics, this mapping of language and problems to symbolic mathematical models.

The examples entirely within mathematics become ever more sophisticated and amazing. Galois essentially gave birth to the field of group theory by his mapping of the problem of solving polynomial equations of arbitrary degree to the problem of finding certain subgroups of symmetry groups associated with the equations. From this linkage he confirmed what others had discovered, that the general polynomial equation of degree 5 does not have a general formula for its solution like the quadratic formula for quadratic equations. Nor in fact do any general equations of degree higher than 5. Only certain equations have such solution formulas, depending on their associated subgroups.

The Laplace Transform maps linear differential equations to rational functions (the ratios of polynomials) where the polynomial variable s corresponds to differentiation and the inverse 1/s corresponds to integration. In my essay on Point Set Topology I indicated how solutions to differential equations can be shown to exist by mapping the set of functions associated with the differential equations to a topological space of points where the solution to a differential equation becomes a fixed point for an associated transformation.

The examples could be multiplied indefinitely. So I feel quite justified in proclaiming that this mapping of physical reality or mathematical problems to some mathematical representation is truly the essence of mathematics. Virtually all mathematical endeavors are involved with some aspect of this activity.

# References

[1] Steckles, Katie, “A couple of mathematical games” (8/15/2017) (https://www.youtube.com/watch?v=VIhyt8pBuMU). James Grime, “Game of Nine” (9/8/2011) (https://www.youtube.com/watch?v=ue4Baa8Mhn8), and “Game of Nine Solution” (9/15/2011) (https://www.youtube.com/watch?v=5I5YVFOJL_g)

These videos all discuss Martin Gardner’s ideas but without reference to him as the originator.

[2] Borges, Jorge Luis, “The Creation and P. H. Gosse,” *Other Inquisitions 1937-1952*, translated by Ruth L. C. Simms, Introduction by James E. Irby, 2nd printing: Clarion Book, Simon and Schuster, New York, 1965, 1st printing: University of Texas Press, 1964.

The gap in our case is 50 years, but the effect is the same. The actual 1937 essay reference refers to Edmond Gosse’s discussion in 1857 of the novel idea of his father, Philip Henry Gosse, concerning creationism. One of Borges’s more delightful essays.

[3] Gardner, Martin (1967), “Jam, Hot, and Other Games”, *Scientific American*, February 1967, pp. 248-266. Also in Martin Gardner, “Jam, Hot, and Other Games”, *Mathematical Carnival*, (1977), The Mathematical Association Of America, Washington, D. C. 1989, Chapter 6, pp.208-212

[4] Wigner, Eugene (1960) “The Unreasonable Effectiveness of Mathematics in the Natural Sciences,” in *Communications in Pure and Applied Mathematics*, vol. 13, No. I (February 1960) (http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html)

This is the classic paper on the mystery of why mathematics so successfully describes physical reality.

[5] Sanderson, Grant, “The most unexpected answer to a counting puzzle” (1/13/2019) (https://www.youtube.com/watch?v=HEfHFsfGXjs), “So why do colliding blocks compute pi” (1/20/2013) (https://www.youtube.com/watch?v=jsYwFizhncE), “How colliding blocks act like a beam of light” (2/3/2019) (https://www.youtube.com/watch?v=brU5yLm9DZM)

[6] Sanderson, Grant, “Sneaky Topology (The Borsuk-Ulam Theorem)” (11/18/2018) (https://www.youtube.com/watch?v=yuVqxCSsE7c)