Counterfeit Coin in Base 3

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Futility Closet presented a method of solving the "counterfeit coin in 12 coins" problem in a way I had not seen before by mapping the problem into numbers in base 3. It wasn't immediately clear to me how their solution worked, so I decided to write up my own explanation.

Bases Into Gold

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(https://www.futilitycloset.com/2018/09/29/bases-into-gold/, retrieved 10/1/2018)

You have 12 coins that appear identical. Eleven have the same weight, but one is either heavier or lighter than the others. How can you identify it, and determine whether it's heavy or light, in just three weighings in a balance scale?

This is a classic puzzle,[1] but in 1992 Washington State University mathematician Calvin T. Long found a solution "that appears little short of magic." Number the coins 1 to 12 and make three weighings:

First weighing: 1 3 5 7 vs. 2 4 6 8
Second weighing: 1 6 8 11 vs. 2 7 9 10
Third weighing: 2 3 8 12 vs. 5 6 9 11

To solve the problem, note the result of each weighing and assemble a three-digit numeral in base 3 as follows:

Left pan sinks: 2
Right pan sinks: 0
Balance: 1

For example, if coin 7 is light, that produces the number 021 in base 3. Now converting that to base 10 gives 7, the number of the odd coin, and an examination of the weighings shows that it must be light. Another example: If coin 2 is heavy, then we get 002 in base 3, which is 2 in base 10. Note that it's possible to get an answer that's higher than 12, e.g. when coin 7 is heavy — in that case subtract the base-10 answer you get from 26.

(Calvin T. Long, "Magic in Base 3," Mathematical Gazette 76:477 [November 1992], 371-376.)

Explanation

First look at the base 3 values for the decimal numbers 1-12 in the first row of Table 1 below, where a decimal number in base 3 is given by

$$N_{base3} = a_2a_1a_0 \text{ where } N_{base10} = a_2\,3^2 + a_1\,3 + a_0 \text{ and } a_0, a_1, \text{and } a_2 = 0, 1, \text{or } 2$$

The second row of the table lists the complements of each base 3 number, that is, the results of swapping the digits 0 and 2 in each base 3 number and leaving 1 unchanged. This operation is equivalent to subtracting each base 3 number from 222, which is 26 in base 10 (decimal). Notice that none of the complements equal the original numbers.

[1] https://www.futilitycloset.com/2013/11/17/counterfeit-redux/

Table 1 Base 3 Numbers and Complements for Decimals 1-12

Coins	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Base3	001	002	010	011	012	020	021	022	100	101	102	110
222 – Base3 (0 ↔ 2)	221	220	212	211	210	202	201	200	122	121	120	112

If there were no counterfeit coin, then all the weighings of 4 coins on each pan would balance and the base 3 number would 111, which is not in the table. If a counterfeit coin was heavy and placed on the left pan for the first weighing and none of the other coins were counterfeit, then the left pan would sink and we would assign 2 to the result. If that heavy counterfeit coin were placed instead on the right pan, then it would sink and we would assign 0 to the weighing. So moving a heavy counterfeit coin from the left to the right pan is equivalent to swapping its first digit from a 2 to a 0. Or moving it from the right pan to the left pan is equivalent to swapping the first digit from a 0 to a 2. If this counterfeit coin were instead light and put on the left pan, then the right pan would sink and the first digit would be 0. Again, putting the light counterfeit coin on the right pan would cause the left pan to sink and the first digit would be 2. And again if the counterfeit coin were not included in the first weighing, then the result would be a 1.

So the key to the problem is to distribute the numbered coins between the right and left pans for the three weighings in such a way that the resulting weighings correspond to a unique base 3 number.

First begin by assuming the counterfeit coin is heavy and put it on the left pan so that if each pan had four coins, the left side would sink on the first weighing. This means the first digit of the number will be 2. So consider the base 3 Table 1 above, second row, and notice that coins 1-8 have a leading digit 2. Provisionally, put them in the left pan (See first row next table). Coins 9-12 have leading digit 1, so they should not appear in the first weighing. The second digit for 6-8 is a 0, so put them on the right pan for the second weighing (second row of the next table), and put coins 1, 2, 9-11 on the left side, since they have second digit 2. Finally, coins 3, 6, 9, 12 have third digit 2, so put them on the left pan for the third weighing (last row of next table), and put coins 2, 5, 8, 11 with third digit 0 on the right pan.

↓				2						0	↓
1	2	3	4	5	6	7	8				
1	2	9	10	11				6	7	8	
	3	6	9	12				2	5	8	11

So right now the distribution of each coin on the left and right corresponds to a unique weighing result that corresponds to that coin if all the other coins are not counterfeit. But clearly, if we actually were to carry out the weighings with the coins distributed as shown in the table, the results would be nonsense because we have an unequal number of coins on either side. So we will move coins to the opposite sides until we have four coins in each pan for each weighing. We will show these swaps do not affect the decimal values of the coins and so the correspondence of weighings to unique coins.

Now, if coin **2** were the (heavy) counterfeit and we move it to the right pan for the first two weighings and to the left pan for the third weighing, that will swap the 0, 2 digits to yield 002 in place of 220, namely, 222-220, which still corresponds to decimal 2 in the base 3 table. Similarly, if coin **4** were the (heavy) counterfeit, moving it to the right pan for the first weighing will change its weighing result from 211 to 011 (it does not appear in the other rows in order to leave a balance and thus a value of 1). Again this base 3 number still corresponds to decimal 4 in the base 3 table. (See next table)

↓				2						0	↓
1	2	3	4	5	6	7	8	2	4		
1	2	9	10	11				2	6	7	8
2	3	6	9	12				2	5	8	11

Continue the swapping with coins **6** and **8**, changing their base 3 values from 202 to 020 and 200 to 022, respectively. (See next table)

↓				2						0	↓
1		3		5	6	7	8	2	4	6	8
1	6	8	9	10	11			2	6	7	8
2	3	6	8	9	12			5	6	8	11

Finally swap coins **9** and **10**, changing their base 3 values from 122 to 100 and 121 to 101, respectively. (See next table)

↓				2						0	↓
1		3		5		7		2	4	6	8
1	6	8	9	10	11			2	7	9	10
2	3	8		9	12			5	6	9	11

Now we have four coins on each pan for each weighing. The weighing results will select the heavy counterfeit coin using the base 3 (or 222 – base 3) assignments in Table 1. If the counterfeit coin were light instead of heavy, then all the 0, 2 digits would swap (0↔2) and the digit 1 remain unchanged. But this would not change the value of the corresponding decimal number and therefore the number of the counterfeit coin.

Thus we have the following sequence of weighings for the 12 coins, which agrees with Calvin T. Long's distribution given initially.

↓		2				0	↓
1	3	5	7	2	4	6	8
1	6	8	11	2	7	9	10
2	3	8	12	5	6	9	11

For example, if the set of three weighings yields the number 220, then from Table 1 we see that the counterfeit coin is #2. Moreover, since #2 is in the right pan during the first weighing and the left pan sinks, we know #2 is therefore light (since the right pan rose).

Or if the three weighings yield the number 012, then we see the base 3 number indicates coin #5 is counterfeit. Since #5 is in the left pan on the first weighing and the right pan sinks, then again we know #5 must be light.