Conical Bottle Problem

(6 November 2018)

Jim Stevenson

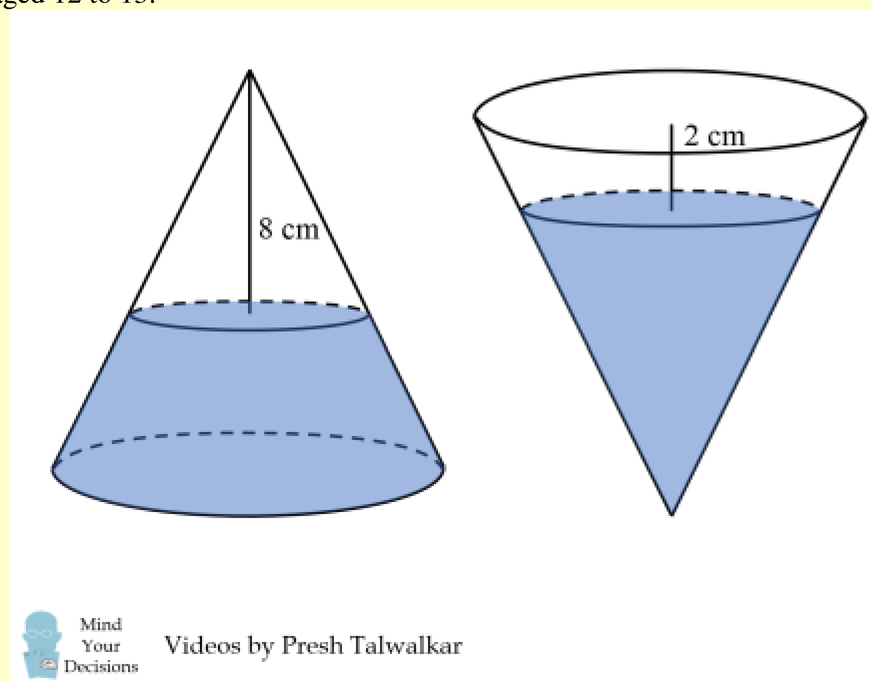
I was astonished that this problem was suitable for 8th graders. First of all the formula for the volume of a cone is one of the least-remembered of formulas, and I certainly never remember it. So my only viable approach was calculus, which is probably not a suitable solution for an 8th grader.

(https://mindyourdecisions.com/blog/2018/11/01/challenge-for-13-year-olds-how-tall-is-the-bottle/, retrieved 11/5/2018)

Challenge for 13 Year Olds – How Tall Is The Bottle?

Presh Talwalkar, November 1, 2018

This was sent to me as a competition problem for 8th graders, so it would be a challenge problem for students aged 12 to 13.



When a conical bottle rests on its flat base, the water in the bottle is 8 cm from its vertex. When the same conical bottle is turned upside down, the water level is 2 cm from its base. What is the height of the bottle? (Note "conical" refers to a right circular cone as is common usage.)

I at first thought this problem was impossible. But it actually can be solved. Give it a try and then watch the video for a solution.

My Solution (Calculus)

Figure 1 shows the set-up for the calculus solution. If the height of the cone is given by the value h and the radius of the base of the cone by the value R, then the cone is given by rotating an inclined

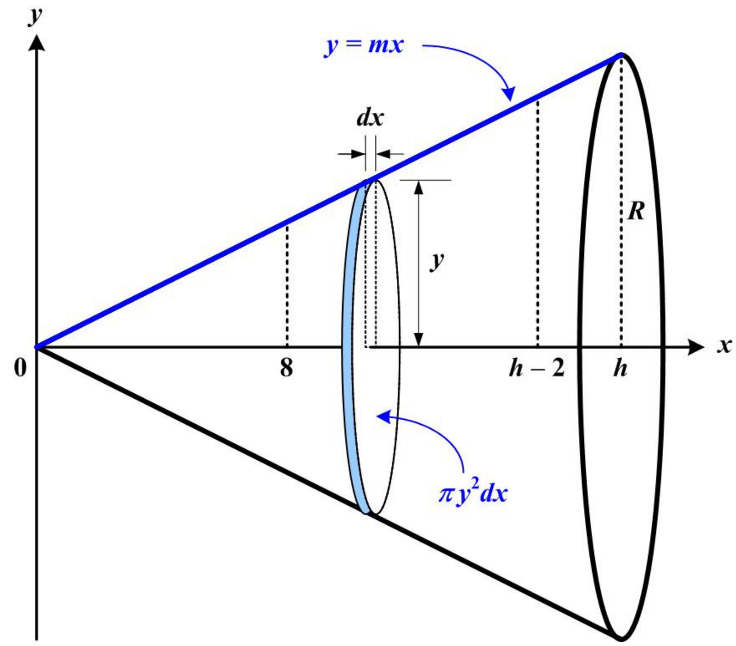


Figure 1 Volume of rotation — Calculus Set-up

line about the x-axis. The line has an equation $y = mx$ where m is the constant R/h. The two volumes of the cone in the problem are obtained by integrating the infinitesimal disks ($\pi y^2\, dx$) between 0 and h - 2 and between 8 and h. Since the volumes are equal, we equate the results. That is,

$$\int_0^{h-2} \pi y^2 dx = \int_8^h \pi y^2 dx$$

or

$$\int_0^{h-2} x^2 dx = \int_8^h x^2 dx$$

since the constant factors πm^2 cancel from both sides. This yields

$$\left.\frac{x^3}{3}\right]_0^{h-2} = \left.\frac{x^3}{3}\right]_8^h$$

or (canceling the $^1/_3$s)

$$(h-2)^3 = h^3 - 8^3 \qquad (*)$$

or

$$h^2 - 2h - 84 = 0$$

Therefore,

$$h = 1 + \sqrt{85}$$

Talwalkar Solution (Geometry)

The key to solving this problem is the volume of water in each cone is the same. Then it becomes a matter of setting up the correct equations and simplifying using similar triangles.

First let's deal with the cone resting on its base (Figure 2). Define the variables:

h – height of bottle
R – radius of bottle's base
r_1 – radius of circle 8 cm from vertex

The volume of water is the volume of the entire bottle minus the volume of the cone 8 cm from the vertex, which is:

$$(\pi/3)R^2h - (\pi/3)(r_1)^2(8)$$

The two right triangles in the diagram are similar, and so we have:

$$r_1/8 = R/h \Rightarrow r_1 = 8R/h$$

We substitute this back into the volume of water formula to get:

$$(\pi/3)R^2h - (\pi/3)(8R/h)^2(8) = (\pi/3)R^2(h - 512/h^2)$$

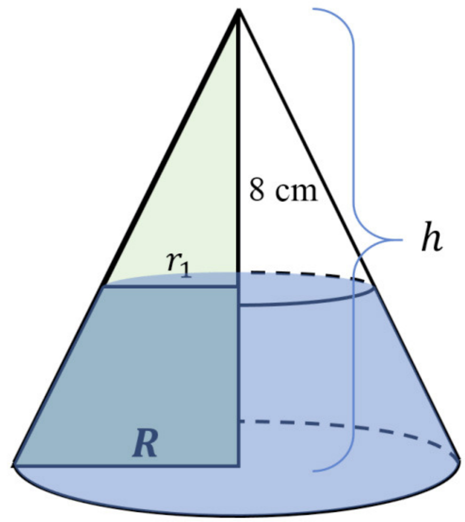


Figure 2

We now do a similar calculation for the other diagram (Figure 3). The height and radius of the bottle is the same, so we only need to define one more variable:

r_2 – radius of circle 2 cm from base

The volume of water is the volume of cone 2 cm from the base, which has a height of $h - 2$. So its volume is:

$$(\pi/3)(r_2)^2(h - 2)$$

Again the two right triangles in the diagram are similar, and so we have:

$$r_2/(h - 2) = R/h \Rightarrow r_2 = (h - 2)R/h$$

We substitute this back into the volume of water formula to get:

$$(\pi/3)((h - 2)R/h)^2(h - 2) = (\pi/3)R^2(h - 2)^3/h^2$$

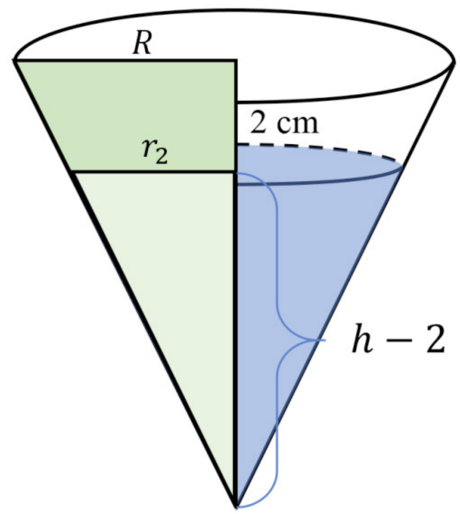


Figure 3

It seems like we have just come up with some complicated formulas. But if we keep working we will find a miraculous cancellation. We now set the two formulas for the volume of water equal to each other.

$$(\pi/3)R^2(h - 512/h^2) = (\pi/3)R^2(h - 2)^3/h^2$$

The term $(\pi/3)R^2$ is common to both sides of the equation, so it cancels out–it turns out the answer is independent of the radius of the bottle!

$$h - 512/h^2 = (h - 2)^3/h^2$$

$$h^3 - 512 = (h - 2)^3 \qquad \text{[= equation (*) above]}$$

$$h^3 - 512 = h^3 - 6h^2 + 12h - 8$$

$$6h^2 - 12h - 504 = 0$$

$$h^2 - 2h - 84 = 0$$

Now we can solve using the quadratic formula, and since height has to be positive, we only keep the solution with $h > 0$:

$$h = 1 + \sqrt{85} \approx 10.2 \text{ cm}$$

And like magic we have solved for the height of the bottle, even without knowing the radius of the bottle!

Interestingly, the answer is very close to 8 + 2 = 10, which would be the case if the water filled exactly half of the bottle. It is a cruel thing they made the "intuitive" answer so close to the actual answer!