Chalkdust Triangle Problem

(13 October 2018, rev 5 March 2018)

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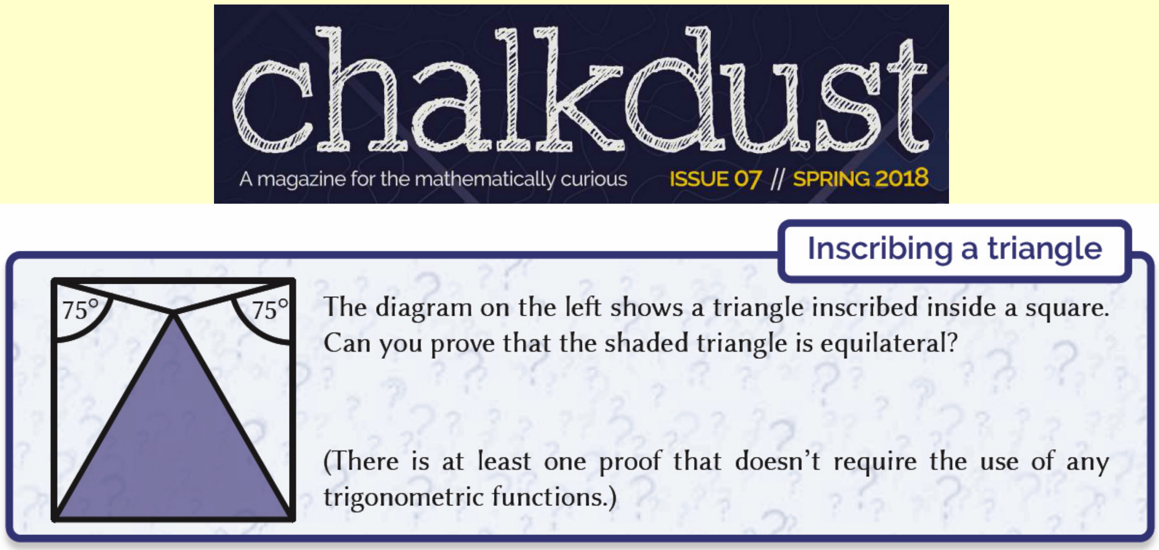
The issue 7 of the *Chalkdust* mathematics magazine had an interesting geometric problem:

(https://drive.google.com/open?id=1FQ5HK2Lw5BLwpeE2ptU0cNj2xIDQpImB, retrieved 10/11/2018)


chalkdust

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Inscribing a triangle

75° 75°

The diagram on the left shows a triangle inscribed inside a square. Can you prove that the shaded triangle is equilateral?

(There is at least one proof that doesn't require the use of any trigonometric functions.)

Solution

There probably is an easier and faster approach,[1] but I had to resort to a number of added constructions to yield a solution. I will give the solution in a number of steps, labeled with circled numbers, such as ①.

Step 1. Figure 1 shows that the base angles in the upside-down triangle are equal (15°) and so the triangle is isosceles. That means the triangles with the 75° angles are congruent via SAS, since their other leg is the side of the square. Hence, the triangle in question is also isosceles. This means the perpendicular from its vertex to its base bisects the angle at the vertex.

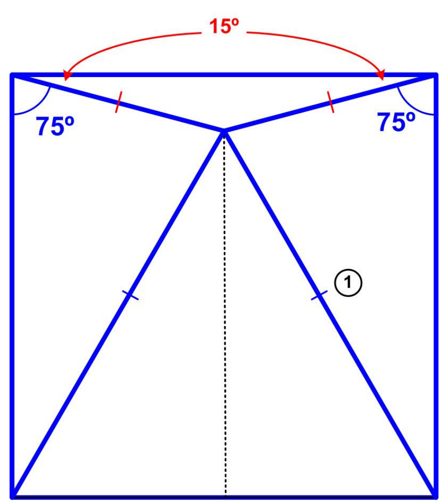


Figure 1 Step 1

Step 2. Now for the added constructions (see Figure 2). Circumscribe the square with a circle. Draw a diagonal of the square which is a diameter of the circle (red dashed line). This diagonal intersects the vertical edge of the square in a 45° angle, cutting the 75° angle into 45° and 30° angles.

[1] There is! See the Chalkdust solution below p.3

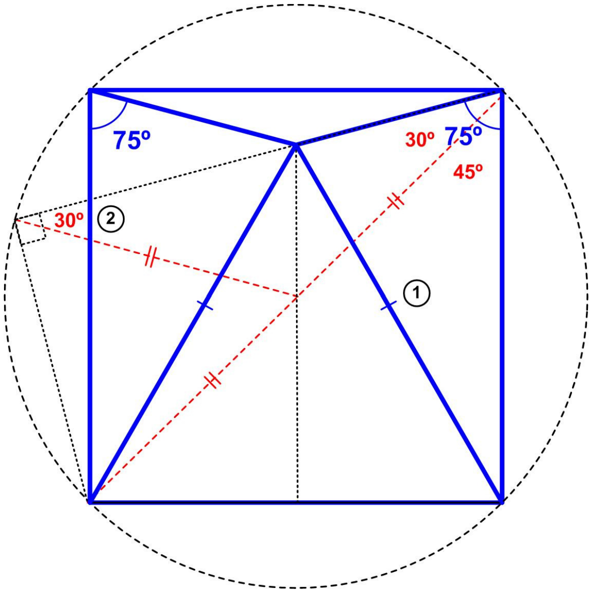


Figure 2 Step 2

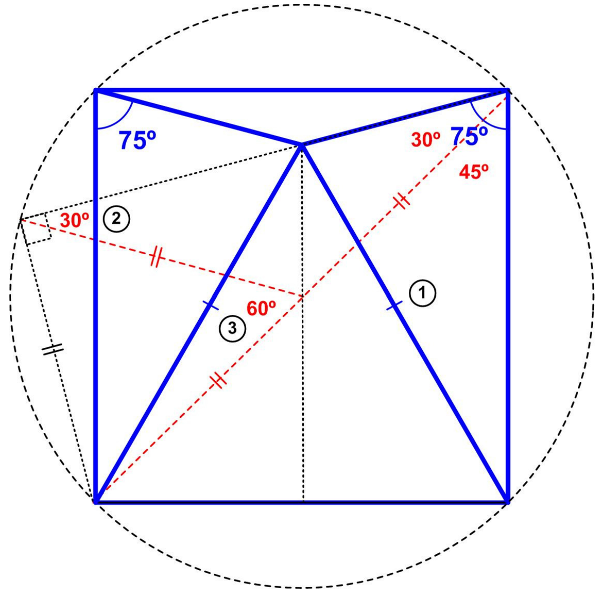


Figure 3 Step 3

Extend the base of the 75° triangle with the 30° angle to intersect the circle and join that intersection to the corresponding corner of the square. (The angle at the intersection is therefore 90°.) Finally draw a radius from the center of the circle to intersection point. Note that all the radii are equal. Therefore the triangle with the 30° angle is isosceles, which means the other angle is also 30°.

Step 3. The exterior angle of the 30° - 30° isosceles triangle is 60° (see Figure 3). The triangle made by the two radii and the short leg of the right triangle is equilateral, since it is isosceles, which implies the base angles are equal and therefore one half of 180° - 60° = 120° or 60°.

Step 4. Since the perpendicular bisector of the triangle in question is parallel to the vertical edge of the square, it also makes an angle of 75° with the extended base of the 75° triangle (see Figure 4). Since 180° - (30° + 75°) = 75°, this triangle is isosceles, so that both legs are equal to the radius.

Step 5. This means the right triangle formed by the two radii and a side of the triangle in question is isosceles (see Figure 5). Hence the base angles are 45°.

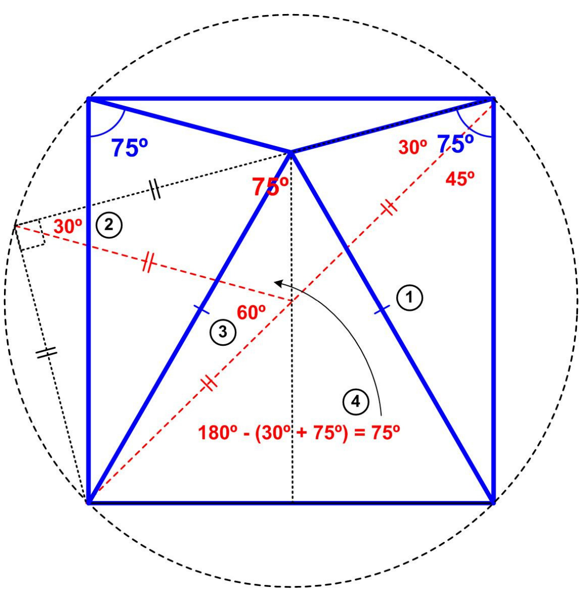


Figure 4 Step 4

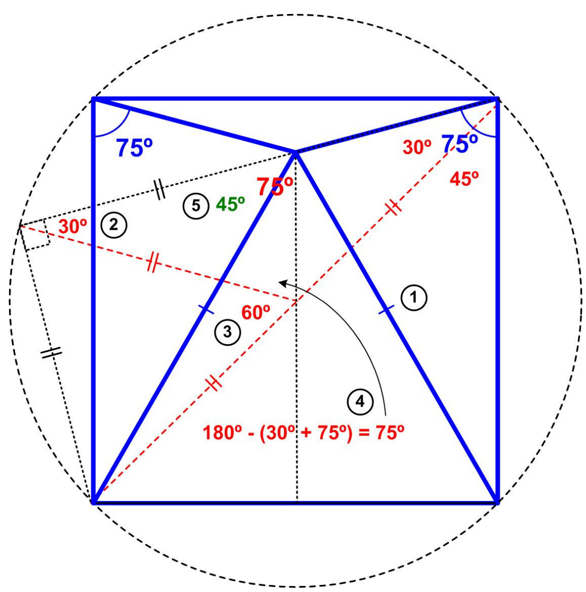


Figure 5 Step 5

Step 6. This means the angle the side of the triangle in question makes with the perpendicular bisector is 30° (see Figure 6). Therefore the vertex angle of the isosceles triangle in question is 60°, which implies that all angles are 60° and so the triangle is equilateral.

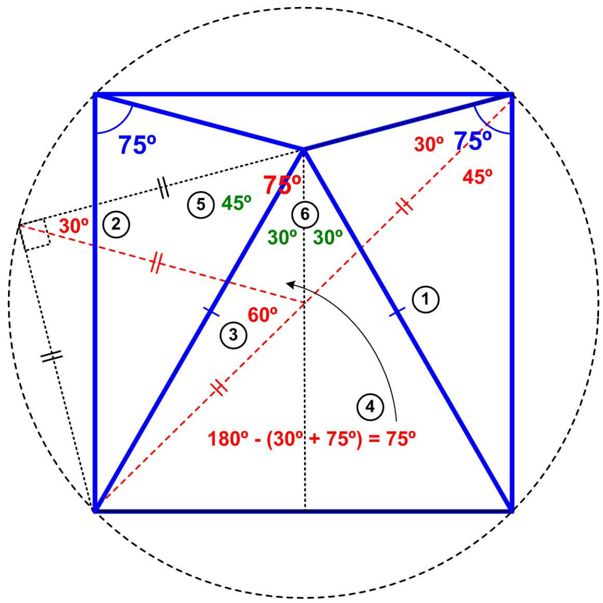


Figure 6 Step 6: Final Solution

Chalkdust Solution

After a bit of searching I found the solution to the problem posed by Mathew Scroggs. It is clearly more elegant than mine and is as follows:

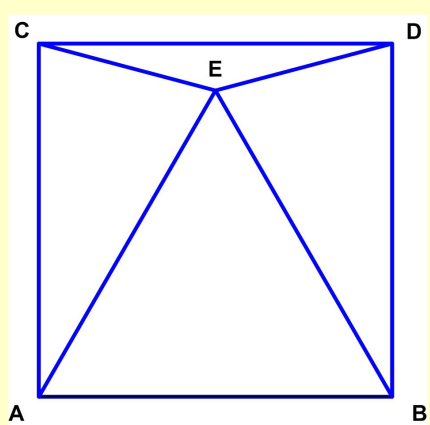
(http://www.mscroggs.co.uk/puzzles/155, retrieved 10/12/2018)

Is It Equilateral?

Matthew Scroggs, March 2018

Source: Chalkdust issue 07 (https://issuu.com/chalkdust/docs/main-smaller/9)

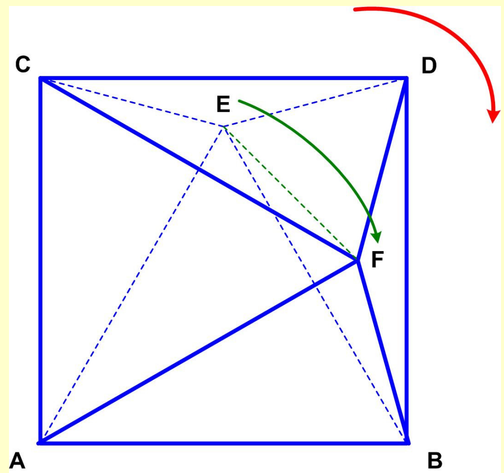
In the diagram below, ABDC is a square. Angles ACE and BDE are both 75°.



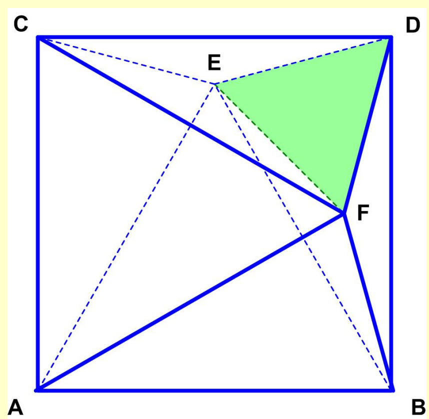
Is triangle ABE equilateral? Why/why not?

Solution

The triangle is equilateral. To see why, add a copy of point E rotated by 90°.[2] This is labelled F on the diagram below.



Angles BDE and CDF are both 75°. Therefore angles CDE and BDF are both 15°. This means that angle FDE is 60°. Line AD is a line of symmetry of the diagram, so angles DFE and DEF are equal and both 60°. Therefore, triangle DEF is equilateral. This triangle is shown in green in the diagram below.



Lines EF, DF and BF are all equal length, so triangles BFE and BFD are isosceles. Angles BDF and FBD are both 15°. Angles FBE and FEB are equal, and the angles in triangle BED add to 180°: this means that angle FBE is 15°. Angles FBE and FBD are both 15°, and so angle EBD is 30°. Angles EBD and ABE add to 90°, and so angle ABE is 60°.

By symmetry, angle BAE is also 60°. Angle BEA must therefore also be 60°, so triangle ABE is equilateral.

Apparently this problem goes back to H. S. M. Coxeter in his text *Introduction to Geometry* (1961), according to Geoff Estes and Richard Moushegian in "The 15° Problem" (7/19/2016) (http://jwilson.coe.uga.edu/EMT668/EMAT6680.F99/Estes/15degrees/15degreeproblem.html). That may be so, but I also found it in Coxeter and Greitzer, *Geometry Revisited*, The Mathematical

[2] At first I could not understand what Scroggs meant here, since you cannot rotate a point. But then I guessed he meant to rotate a copy of the entire figure 90° clockwise, which would move point E to point F. Then add line EF to the resulting figure. (I replaced the original figures with my own.)

Association of America, 1967, Chapter 1 Points and Lines Connected with a Triangle, §1.9 Pedal Triangles, Exercise #2, p.25:

> If an isosceles triangle PAB, with equal angles 15° at the ends of its base AB, is drawn inside a square ABCD, as in Figure 1.9C, then the points P, C, D are the vertices of an equilateral triangle.
>
> **Solution** (p.168): First relax the conditions by allowing $ABCD$ to be a rectangle. Suppose, if possible, that $PD < CD$. then $\angle CPD > 60°$, $\angle DPA < 75°$, $AD < PD < CD$. If, on the other hand, $PD > CD$, all the inequalities are reversed. In either case $ABCD$ would not be a square. Hence, if $ABCD$ is a square, we must have $PD = CD$.
>
> **Or**: Construct $\Delta BQC \cong \Delta APB$ (see Fig. 1.9C). Then ΔBPQ is equilateral, CQ extended is perpendicular to PB and bisects it, and $CP = CB = CD$. Similarly, $DP = DC$.

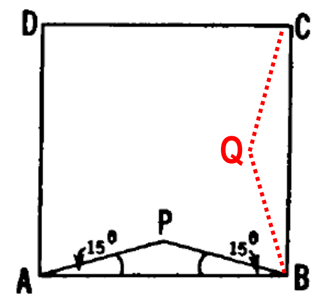


Figure 1.9C