Lorentz Transformation

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by Jim Stevenson

Over the years one of the subjects I return to periodically to study is Einstein's Theory of Relativity, both the Special and General theories. Interest in the Special Theory focused on the derivation of the Lorentz transformations (or contractions). Why did objects appear with different lengths and clocks run at different speeds for observers moving relative to one another? Early on (late 60s) I came across a great explanation in the 1923 book by C. P. Steinmetz ([1] pp.24-27):

THE FOUR-DIMENSIONAL TIME-SPACE OF MINKOWSKI

The relativity theory shows that length is not a constant property of a body but depends on the conditions under which it is observed. This does not mean that a body, like the railway train of our previous instance, has at some time one length, l_0, and at another time another length, l_1, but it means that at the same time the railway train has different lengths to different observers. It has the length l_0 to one observer—for instance, the observer in the railway train, who is at rest with regard to it—and at the same time a different (and shorter) length, l_1, to another observer—for instance, the observer standing near the track and watching the train passing by—and it would have still another length, l_2, to a third observer having a different relative speed with regard to the train. The same applies to the time. That is, the beat of the second-pendulum in the train has the duration t_0 to the observer in the train, and the same beat of the same second-pendulum in the train has a different (and longer) duration, t_1, to an observer on the track; and so on.

Thus the length of an object depends on the velocity of its relative motion to the observer, and as velocity is length divided by time, this makes the length of an object dependent on the time. Inversely, as the time depends on the velocity of the relative motion, the time depends on length. Thus length—that is, space dimension—and time become dependent upon each other.

We always have known that this world of ours is in reality four-dimensional—that is, every point event in the world is given by four numerical values, data, coordinates or dimensions, whatever we may call them, three dimensions in space and one dimension in time. But because in the physics before Einstein space and time were always independent of each other, we never realized this or found any object or advantage in considering the world as four-dimensional, but always considered the point events as three-dimensional in space and one-dimensional in time, treating time and space as separate and incompatible entities. The relativity theory, by interrelating space and time, thus changes our entire world conception.

The dependence of length and time on the relative velocity and thus on each other is an inevitable conclusion from the relativity theory—that is, from the two assumptions.

(1) That all motion is relative, the motion of the railway train relative to the track being the same as the motion of the track relative to the train, and

(2) that the laws of nature, and thus the velocity of light, are the same everywhere.

Consider, in Fig. 1; our illustration of a railway train R, moving with the velocity v, for instance, at 60 miles per hour, relative to the track B. Let us denote the distance relative to the train—that is, measured in the train—by x', and the time in the train by t'. The distance measured along the track may be denoted by x and the time on the track by t. For simplicity we may count distance and time, in the train and on the track, from the same zero value—that is,

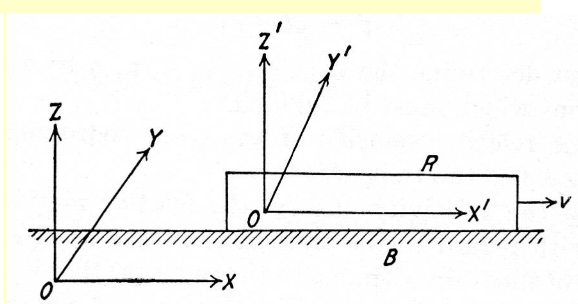

Fig. 1

assume x = 0, t = 0, x' = 0, t' = 0. (This obviously makes no essential difference, but merely eliminates unnecessary constant terms in the equations of transformation from train to track and inversely.)

x' and t', the coördinates with regard to the train, thus are moving at velocity v relative to the coordinates x and t with regard to the track, and by the conventional or Newtonian mechanics, we would have:

t' = t,

that is, the time is the same on the track and in the train, and

x = x'+ vt',

that is, the distance along the track x of a point of the train increases during the time t by vt', that is, with the velocity v. These equations do not apply any more in the relativity theory as they would give different velocities of light relative to the train and relative to the track. To find the equations which apply, we start with the most general relations between x, t and x', t', that is:'[1]

x' = ax – bt, (1)

t' = pt – qx,

and then determine the constants a, b, p, q by the **three conditions** which must be fulfilled.

1. The relative velocity of the train coördinates x', t' with regard to the track x, t is v.
2. By the relativity theory, the relative motion of the track with regard to the train is the same as the relative motion of the train with regard to the track; that is, x', t' are related to x, t by the same equations as x, t are related to x', t' [except v is changed to –v].
3. The velocity of light *c* on the track, in the x, t coördinates, is the same as in the train, in the x', t' coordinates.

These three conditions give four equations between the four constants a, b, p, q, and thereby determine these constants …

Steinmetz then gives a derivation that I did not follow completely, so I produced my own, which I will give here. The thing that impressed me was how simple and transparent the three conditions were. They certainly captured the essence of special relativity. And to have the Lorentz contractions result from just these premises made them all the more believable.

One caveat about the Steinmetz derivation is his assumption that the equations relating the two coordinates are linear. This is not obvious. Taylor and Wheeler (1992 [3] p.100) address this and establish the linearity.

My Derivation

Let's designate the coordinate system on the train by S' and the one on the ground by S. Then assuming linearity, the change in coordinates is given by the linear equations

$$x' = ax + bt \qquad (2)$$

$$t' = px + qt$$

where a, b, p, and q are constants independent of x and t. It doesn't really matter, but contrary to Steinmetz' equation (1), in order to appear a bit more general I have not included minus signs in the equations at this point and I have not reversed the order of x and t in the second equation.

[1] The relation must be linear, as it is univalent. [Footnote in the original, but it is not the correct explanation]

The real import of Steinmetz' remark about "univalent" is that it implies equations (2) are invertible, that is, we can express x and t in terms of x' and t'. This is because there is a one-to-one correspondence (1–1) between the coordinate pairs (x, t) and (x', t'). To each (x, t) there corresponds one and only one (x', t') and vice versa, because they both represent the same physical point and distinct coordinates represent distinct points. This 1–1 property constrains the constants in equation (2) so that $aq - bp \neq 0$. This expression is called the determinant of the linear equations and designated Δ. The inverse equations are then given by

$$x = \frac{1}{\Delta}(qx' - bt') \tag{3}$$

$$t = \frac{1}{\Delta}(-px' + at')$$

which can be found by Cramer's rule or just taking equations (2) and solving for x and t in terms of x' and t'.

Since the origin in the train frame S' moves with velocity v relative to the ground frame S, every point of the form (0, t') in S' corresponds to the point (x, t) in S where x = vt, that is,

$0 = a(vt) + bt$

$t' = p(vt) + qt$

so that $0 = av + b \Rightarrow b = -av$. Conversely, the origin in S moves with velocity –v in S'. Therefore, every point (0, t) in S corresponds to (x', t') in S' such that x' = –vt', that is, from equation (3),

$$0 = \frac{1}{\Delta}(-qvt' - bt')$$

$$t = \frac{1}{\Delta}(pvt' + at')$$

Therefore, $0 = -qvt' - bt' \Rightarrow b = -qv \Rightarrow a = q$.

Now

$$\frac{x'}{t'} = \frac{ax + bt}{px + qt} = \frac{a\frac{x}{t} + b}{p\frac{x}{t} + q} \tag{4}$$

This equation relates the (constant) velocity x'/t' of a particle in S' with its velocity x/t in S. Since the speed of light is the same in both frames S and S' (according to condition 3), x/t = c if and only if x'/t' = c. Thus from equation (4) we have

$$c = \frac{ac + b}{pc + q}$$

Therefore $pc^2 + cq = ac + b = qc - av$ (since a = q and b = –av). And so $p = -av/c^2$.

Now equations (2) and (3) become

$$x' = ax - avt \qquad \text{and} \qquad x = \frac{1}{\Delta}(ax' + avt')$$

$$t' = \frac{-av}{c^2}x + at \qquad\qquad t = \frac{1}{\Delta}\left(\frac{av}{c^2}x' + at'\right)$$

From Steinmetz Condition 2, equations (2) and (3) should take the same form except for v in (2) becoming –v in (3). Then the determinant Δ must be 1. Therefore $1 = \Delta = aq - pb = a^2 - a^2v^2/c^2 \Rightarrow$

$$a = \pm\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}.$$

If we want for x > 0 that (x, 0) corresponds to (x', t') with x' > 0, the we must choose a > 0. Thus we arrive at the **Lorentz transformations**

$$x' = \frac{1}{\sqrt{1-\beta^2}}x - \frac{\beta}{\sqrt{1-\beta^2}}ct \tag{5}$$

$$ct' = -\frac{\beta}{\sqrt{1-\beta^2}}x + \frac{1}{\sqrt{1-\beta^2}}ct$$

where β = v/c.

Remarks

After some tedious computations using (5), it can be shown that

$$c^2t^2 - x^2 = c^2t'^2 - x'^2 \tag{6}$$

This expression is called the *interval* and equation (6) shows it is invariant under the Lorentz Transformation of coordinates. Taylor and Wheeler (1966 [2] and 1992 [3]) proceed in the opposite direction from our approach to derive the Lorentz Transformation from the assumption of the invariance of the interval.

The form of equations (5) also suggests that changing the units of time from seconds, say, to the distance traveled at the speed of light for those seconds would simplify the expressions. This is what Taylor and Wheeler ([2] p.17) do:

> … we discussed how to obtain such a calibration by bouncing a flash of light back and forth between two mirrors one-half meter apart. This mirror clock is said to "tick" each time the light flash arrives back at the first mirror. Between ticks the light flash travels a round-trip distance of 1 meter: Call the unit of time between ticks *1 meter of light-travel time*, or more simply, *1 meter of time*.

Alternatively, we could divide the expressions in equations (5) by the speed of light c and then have distance given in units of time. This is in fact what we do when we say some astronomical system is so many light-years away: it is the distance traveled by the speed of light over the given amount of time.

References

[1] Steinmetz, Charles Proteus, *Four Lectures on Relativity and Space*, McGraw Hill, 1923, reprinted by Dover Publications, New York, 1967. pp.24-26

[2] Taylor, Edwin F. and John Archibald Wheeler, *Spacetime Physics*, W. H. Freeman and Co., 1966

[3] Taylor, Edwin F. and John Archibald Wheeler, *Spacetime Physics: Introduction to Special Relativity*, 2d Ed., W. H. Freeman and Co., 1992