Bugles, Trumpets, and Beltrami

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This essay began as an effort to prove Tanya Khovanova's statement in her article "The Annoyance of Hyperbolic Surfaces" ([1]) that her crocheted hyperbolic surface had constant (negative) curvature. I discussed Khovanova's article in my previous essay "Exponential Yarn" ([5]). What I thought would be a fairly straight-forward exercise turned into a more concerted effort as I concluded that her crocheted surface did not have constant curvature. However, I found additional references ([8], [9]) that supported her statement, so I was becoming quite confused. I looked at other, similar surfaces to try to understand the whole curvature situation. This involved a lot of tedious computations (with my usual plethora of mistakes) that proved most challenging. But then I realized where I had gone astray. To cover my ignorance I claimed my error stemmed from a *subtle* misunderstanding. Herewith is a presentation of what I found.

Curvature

First I need to explain simply what curvature in a surface is and how it can be computed. I will leave the derivations to the text books. I relied heavily on my old 1966 basic differential geometry book by O'Neill ([2]). I compared it with the more recent 2006 2nd edition ([3]), and I also referred to the very fine 2006 book by Wolfgang Kühnel ([4]). I have discussed curvature before in my article "Degree of Latitude"([6]).

To define the curvature of a surface at a point P (Figure 1), first consider the (unit) normal vector **N** to the surface at that point. It is the vector perpendicular to the tangent plane at the point. There are two choices for the direction of the normal. We just pick one. Now pass a plane through the normal cutting the surface in a curve through P. This defines a curve in the cutting plane. Find the osculating (kissing) circle to the curve at the point P in the cutting plane. The circle has the same tangent **T** (basically 1st derivative, velocity) and concavity (basically 2nd derivative, acceleration) as the curve. So it can be thought of as the "closest" circle one can find to the curve at the point P. This means the circle and the curve have the same "curvature" in the calculus sense at P.

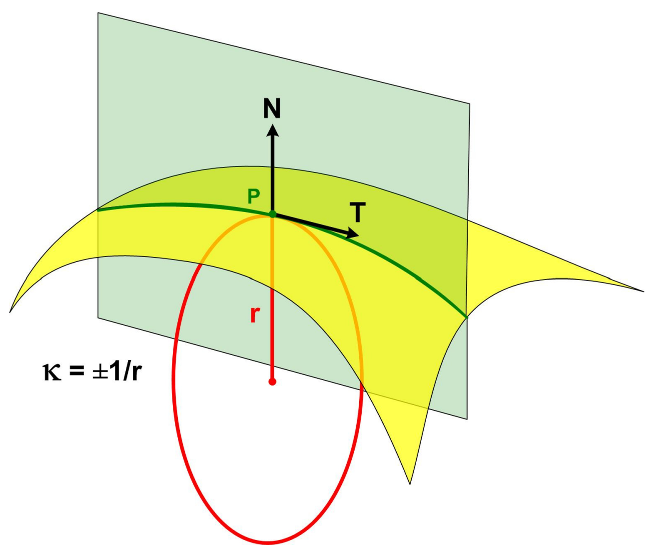

Figure 1 Definition of Normal Curvature of a Surface at a Point

Furthermore, the curvature of the circle is constant and equal to the reciprocal of its radius. (We have skirted some details such as both the circle and curve must be parameterized by arclength.) Recall that for an object moving with a constant speed v around the circle, its acceleration is radially toward the center and given by $a = v^2/r$ where r is the radius of the circle. So if the speed of the object is unit speed (the position of the object along the circle is parameterized by its arclength (segment of circumference) from some fixed point), then v = 1 and the curvature becomes 1/r.

Therefore we can define the ***normal curvature*** *of the surface at the point P in the direction* ***T*** as **κ = ±1/r**, where r is the radius of the osculating circle and called the *radius of curvature.* The sign of the curvature is given by the direction the tangent **T** is turning, that is, if **T** or the curve is turning towards the normal **N**, then the sign is positive, and if it is turning away from **N**, then the sign is negative.

Clearly, pivoting the surface-cutting plane about the normal will produce different curves in different directions and therefore different normal curvatures. Unless the surface is like a sphere where the normal curvature in any direction at the point P is the same (P is called an ***umbilic point***), the normal curvature will vary from a maximum to a minimum (and the corresponding radius of curvature from a minimum to a maximum). These maximum and minimum curvatures are called the ***principal curvatures*** of the surface at P and denoted k_1 and k_2. Furthermore, the directions giving these maxima or minima are called ***principal directions*** and the corresponding unit tangent vectors the ***principal vectors*** of the surface at P. It turns out that *these principal vectors are always orthogonal* to each other.

We come now to the final main definition: The Gaussian curvature K of a surface at a point P is the product of the principal curvatures at P, that is,

$$\textbf{(Gaussian Curvature)} \qquad K = k_1 \cdot k_2$$

Notice that we also have $K = (1/r_1)(1/r_2) = (1/r_1r_2)$ where r_1 and r_2 are the radii of curvature of the corresponding principal curvatures. The significant thing about the Gaussian curvature is that it is intrinsic, that is, it is what someone living on the surface with no understanding of an embedding in a higher dimension would calculate.

For example, for the surface shown in Figure 1, the tangents **T** will always be turning away from the direction of the normal **N**, so both principal curvatures k_1 and k_2 are negative, their product is positive, and so the resulting Gaussian curvature is positive.

Surface of Revolution

We shall now consider various surfaces of revolution of the form shown in Figure 2. They constitute a profile curve or generator rotated around an axis (the x-axis in our case). The curves represented by the images of the profile curve on the surface are called *meridians* and the circles generated by points on the profile curve rotated around the axis are called *parallels*. Points on the surface can be parameterized by

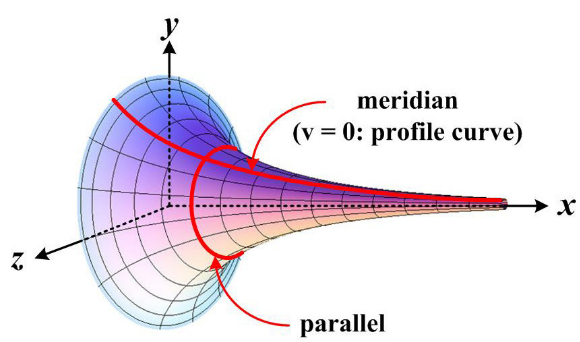

Figure 2 Surface of Revolution
(https://plus.google.com/+YamadaseikiCoJpGeometricModel/posts/ea2oEXJQGhg)

$$\psi(u, v) = (x, y, z) = (g(u), h(u) \cos v, h(u) \sin v) \quad (1)$$

where $u \in [0, \infty)$, $v \in [0, 2\pi)$ and where $x = g(u)$ is the distance along the x-axis and h(u) is the radius of the parallel circle at that point. For any fixed value of v (in particular v = 0), the meridian can be parameterized as

$$\alpha(u) = (g(u), h(u) \cos v, h(u) \sin v)$$

$$\alpha(u) = (g(u), h(u), 0) \qquad \text{(profile curve)} \qquad (2)$$

And for any fixed u, the parallel circle can be parameterized as

$$\beta(v) = (g(u), h(u) \cos v, h(u) \sin v)$$

Furthermore, the derivative (velocity vector) of the meridian is given by

$$\alpha'(u) = \psi_u(u, v) = (g'(u), h'(u) \cos v, h'(u) \sin v)$$

and the derivative (velocity vector) of the parallel is given by

$$\beta'(v) = \psi_v(u, v) = (0, -h(u) \sin v, h(u) \cos v)$$

Then the dot product of vectors $\alpha'(u) \cdot \beta'(v) = 0$ implies they are orthogonal and the meridians and parallels are perpendicular to each other at each point of the surface (Figure 3, based on figures from [7]).

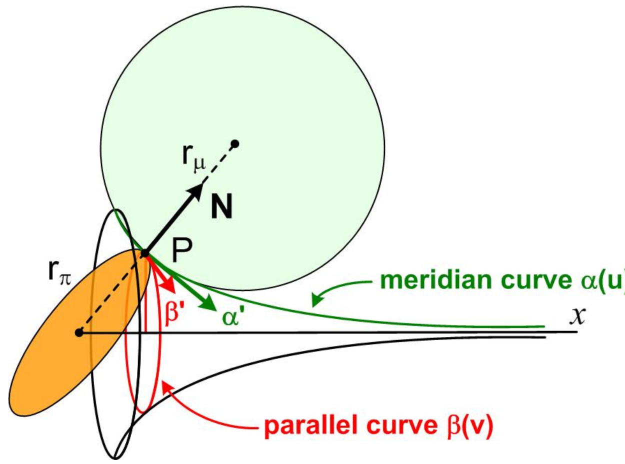

Figure 3 Surface of Revolution with Curvature (Principal) Directions

It should be evident (and can be calculated) that the tangent vector $\alpha'(u)$ is pointing in a principal direction, that is, that the normal curvature at a point on the profile curve is maximal (near the origin) and minimal (out along the x-axis). That means that the vector $\beta'(v)$ tangent to the parallel at the same point and orthogonal to the profile tangent $\alpha'(u)$ must point in the other principal direction. So the unit versions of these tangent vectors are the principal vectors at each point of the surface of revolution. Curves in a surface whose tangent vectors *all* point in principal directions are called ***principal curves***. So the meridians and parallels of a surface of revolution are principal curves.

This terminology confused me for some time (and cost me several days of back and forth computations). *The curvature of the parallel principal curve is not a principal curvature!* As you can see in Figure 3, the parallel curve $\beta(v)$ does *not* lie in a plane through the unit normal **N**; it only shares the unit tangent vector (principal vector) with the osculating circle that does lie in the plane. Figure 4 shows the relationship of the curvature of a curve β whose tangent points in a principal direction. If $\mathbf{N}(\beta)$ represents a unit normal to this curve at some point P in the surface, then the normal (principal) curvature k_π at P is related to the (Frenét) curvature κ of β by ([2] p.197)

$$k_\pi = \kappa\, \mathbf{N}(\beta) \cdot \mathbf{N} = \kappa \cos\theta.$$

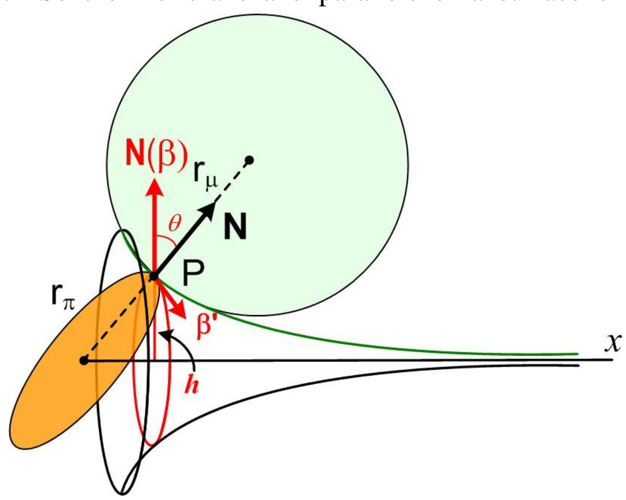

Figure 4 β Curve Curvature κ vs Principal Curvature k_π

Since $\kappa = 1/h$ is the curvature of the parallel circle, the principal curvature

$$k_\pi = 1/r_\pi = \cos\theta / h \;\Rightarrow\; h = r_\pi \cos\theta .$$

This means the principal radius of curvature in the parallel direction is the hypotenuse of the right triangle with h as one leg and a segment of the x-axis as the other. That means the center of the osculating circle lies on the x-axis, as shown in Figure 3 and Figure 4.

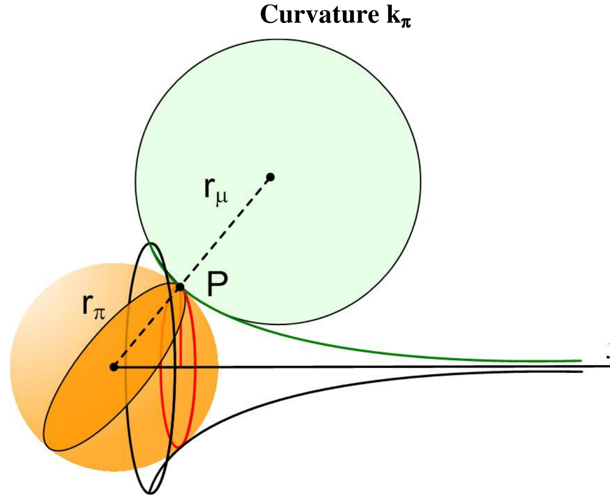

Figure 5 Radius of Curvature r_π Sphere

Also notice that radii of curvature r_π and

r_μ are on opposite sides of the tangent plane. This means their corresponding curvatures will have opposite signs. Therefore the product of these curvatures, the Gaussian curvature, will be negative.

By the rotational symmetry of the surface of revolution, we see that the osculating circle sweeps out a sphere centered on the x-axis (Figure 5). This means for diagrammatic purposes we can consider the osculating circle for r_π in the same plane as the one for r_μ (Figure 6).

We are now ready to consider different profile curves and the different curvatures of the corresponding surfaces of revolution.

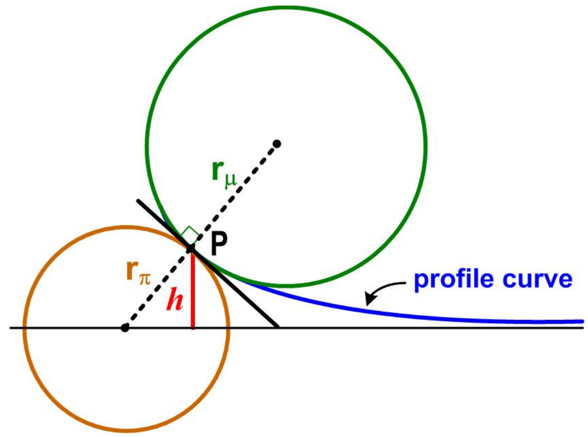

Figure 6 2-D Scheme for Computations

Examples of Surfaces of Revolution

We now need to compute some explicit curvatures for several surfaces of revolution. I will use the formulas and notation from O'Neill ([2] p.234, [3] p.253). Using the parameterized version of the profile curve in equation (2), $\alpha(u) = (g(u), h(u), 0)$, we get the principal curvatures in the direction of the meridian and parallel, along with the Gaussian curvature K, as:

$$k_\mu = \frac{-\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{\left(g'^2 + h'^2\right)^{3/2}}, \qquad k_\pi = \frac{g'}{h\left(g'^2 + h'^2\right)^{1/2}}, \qquad K = k_\mu k_\pi = \frac{-g'\begin{vmatrix} g' & h' \\ g'' & h'' \end{vmatrix}}{h\left(g'^2 + h'^2\right)^2} \tag{3}$$

In the case where g(u) = u, we get:

$$k_\mu = \frac{-h''}{\left(1 + h'^2\right)^{3/2}}, \qquad k_\pi = \frac{1}{h\left(1 + h'^2\right)^{1/2}}, \qquad K = \frac{-h''}{h\left(1 + h'^2\right)^2} \tag{4}$$

Hyperbola Profile – Torelli's Trumpet

The first example is included because it is a standard example from calculus and has non-constant negative curvature. The profile curve is $\alpha(u) = (u, h(u), 0) = (u, 1/u, 0)$ for $0 < u < \infty$. The surface of revolution is colorfully called Torelli's Trumpet.

The surface area and volume of Torelli's Trumpet are computed from improper integrals, usually from some fixed point a > 0, say a = 1, over the whole positive real line. This entails taking the limits of finite integrals over the closed intervals [a, b] where $b \to \infty$. It turns out that the improper integral for the surface area does not converge (have a limit as $b \to \infty$), but the improper integral for the volume does converge. Therefore one jokes that to paint the trumpet's surface one must fill it with paint instead. Thus the usual weird things happen when infinity is involved.

Using equations (4), we obtain the principal curvatures and Gaussian curvature

$$k_\mu = \frac{-2u^3}{\left(1 + u^4\right)^{3/2}}, \qquad k_\pi = \frac{u^3}{\left(1 + u^4\right)^{1/2}}, \qquad K = \frac{-2u^6}{\left(1 + u^4\right)^2} = -\frac{1}{\frac{1}{2}\left(u + \frac{1}{u^3}\right)^2}$$

(See Figure 7). Notice that the radius of curvature of the parallel $r_\pi = 1/k_\pi = (1/u^2 + 1/u^6)^{1/2}$ agrees

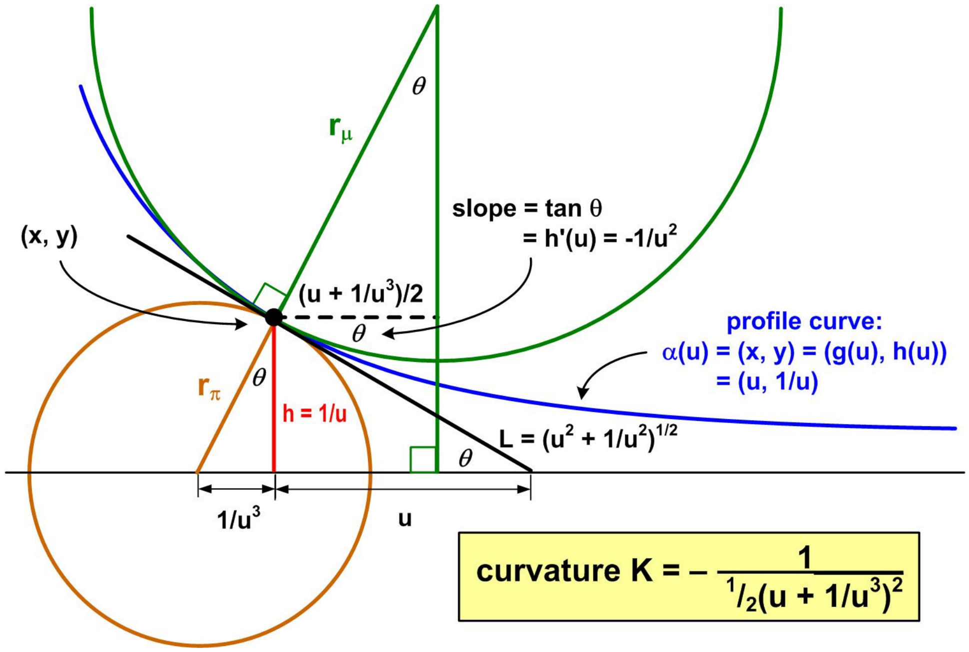

Figure 7 Torelli's Trumpet (hyperbolic profile curve) with Gaussian curvature K and radii of curvature r_π, r_μ

with the argument given above illustrated in Figure 4 where r_π is shown to be the hypotenuse of the triangle with leg h(u) = 1/u. Notice that the angle θ in Figure 7 is the same as the angle θ in Figure 4. tan θ represents the slope of the tangent to the profile curve or h'(u), and so equals $1/u^2$ (ignoring signs). This implies the segment of the x-axis equals $1/u^3$ and so agrees with the value for r_π via the Pythagorean Theorem.

Other things to notice are (1) the length of the tangent line to the x-axis L varies and (2) the x-distance of the tangent point (x, y) to the center of the osculating circle with radius r_μ is $r_\mu \sin\theta = (u + 1/u^3)/2$.

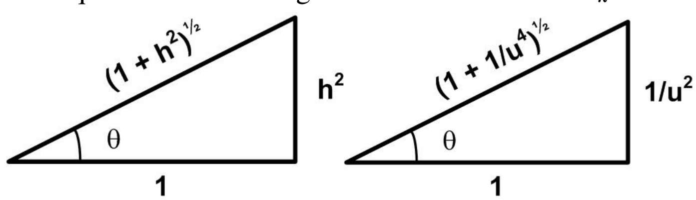

Figure 8 Trigonometric Relations

Exponential Profile

We turn now to the example that instigated this investigation: my representation of Khovanova's crocheted "surface" as a surface of revolution with a profile curve of the form $R2^x$. First, we will convert this expression of doubling into the more customary form using the exponential function $y = e^x$ via $R2^x = Re^{\ln 2\, x}$. This expression is an example of the general solution to the exponential differential equation dy/dx = kx given by $y = y_0 e^{kx}$ where $y = y_0$ when x = 0. In other words y is the general quantity that grows proportionally to its current value at any instant. For simplicity, we shall set all constants equal to 1. Furthermore, we shall reverse the direction of Khovanova surface to conform to the surface of revolution shapes we have been considering, namely, we shall consider $y = e^{-x}$.

So the parameterization of the profile curve we are interested in is $\alpha(u) = (u, h(u), 0) = (u, e^{-u}, 0)$ for $0 \le u < \infty$. Since h' = -h and h" = h, this yields the following principle curvatures and Gaussian curvature:

$$k_\mu = \frac{-h}{\left(1+h^2\right)^{3/2}}, \qquad k_\pi = \frac{1}{h\left(1+h^2\right)^{1/2}}, \quad K = -\frac{h}{\left(1+h^2\right)^2} = -\frac{e^{-u}}{\left(1+e^{-2u}\right)^2}$$

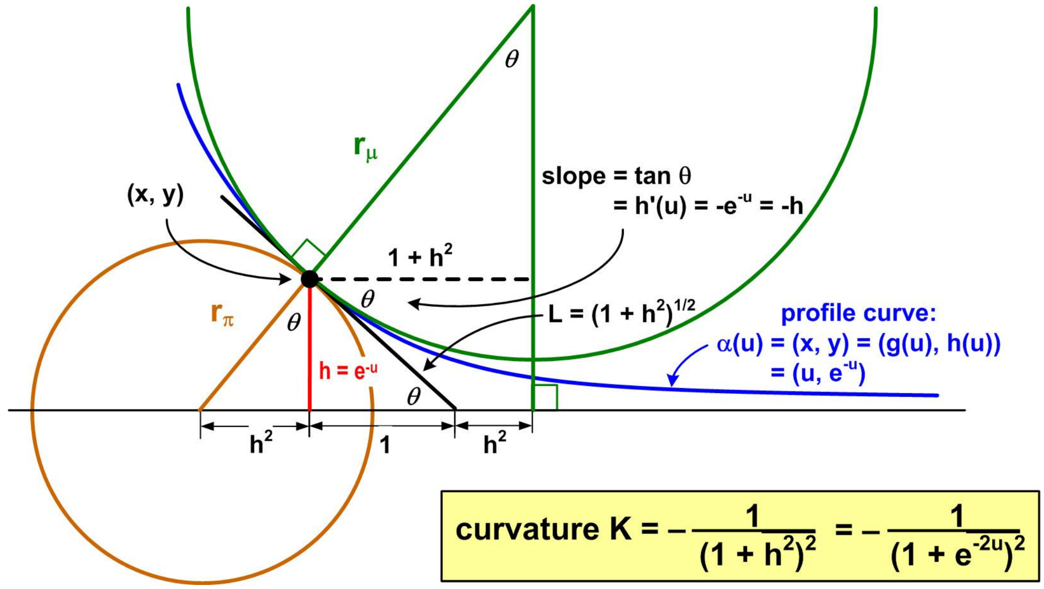

Figure 9 Exponential profile curve with Gaussian curvature K and radii of curvature r_π, r_μ

(See Figure 9) Again, the Gaussian curvature K is negative and not constant: it ranges from -1/4 to 0 as u goes from 0 to ∞. Again the radius of curvature of the parallel $r_\pi = 1/k_\pi = h(1 + h^2)^{1/2}$ agrees with the argument given above illustrated in Figure 4 where r_π is shown to be the hypotenuse of the triangle with legs h and h^2 ($h^2 = h \tan \theta$).

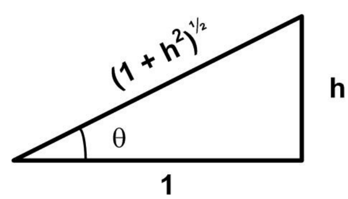

Figure 10 Trigonometric Relations

As with Torelli's Trumpet the length of the tangent line to the x-axis $L = (1 + h^2)^{1/2}$ varies and the x-distance of the tangent point (x, y) to the center of the osculating circle with radius r_μ is $r_\mu \sin \theta = 1 + h^2$.

So at this point I was perplexed at how this surface of revolution generated by the exponential function could be interpreted to have *constant* negative curvature.

Tractrix Profile – Tractroid, Bugle, Beltrami Surface

The surface of revolution that *does* have constant negative curvature is called a bugle surface or Beltrami surface, or finally a tractroid because its profile curve is a tractrix. The tractrix has the colorful description of being the track made by a wagon being pulled by a child walking along a straight line, or the rear wheel of a tractor trailer making a right-angle turn (Figure 11).

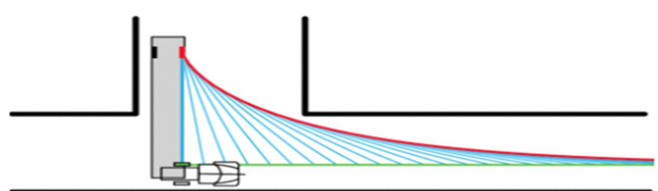

Figure 11 Tractrix from Turning Tractor Trailer
(http://www.qedcat.com/archive_cleaned/158.html)

The key idea is that the handle of the wagon or the side of the trailer are constant in length and tangent to the tractrix. That is, the distance L along the tangent line from the point of tangency (u, h(u)) to the x-axis is a constant c for this curve (see Figure 12). This means the slope of the tangent line or derivative $h'(u) = -h / (c^2 - h^2)^{1/2}$ and therefore $h''(u) = -c^2h' / (c^2 - h^2)^{3/2}$. These relations yield the following principle curvatures and Gaussian curvature:

$$k_\mu = \frac{-h}{c\left(c^2-h^2\right)^{1/2}} = \frac{h'}{c}, \qquad k_\pi = \frac{\left(c^2-h^2\right)^{1/2}}{ch} = -\frac{1}{ch'}, \qquad K = -\frac{1}{c^2}$$

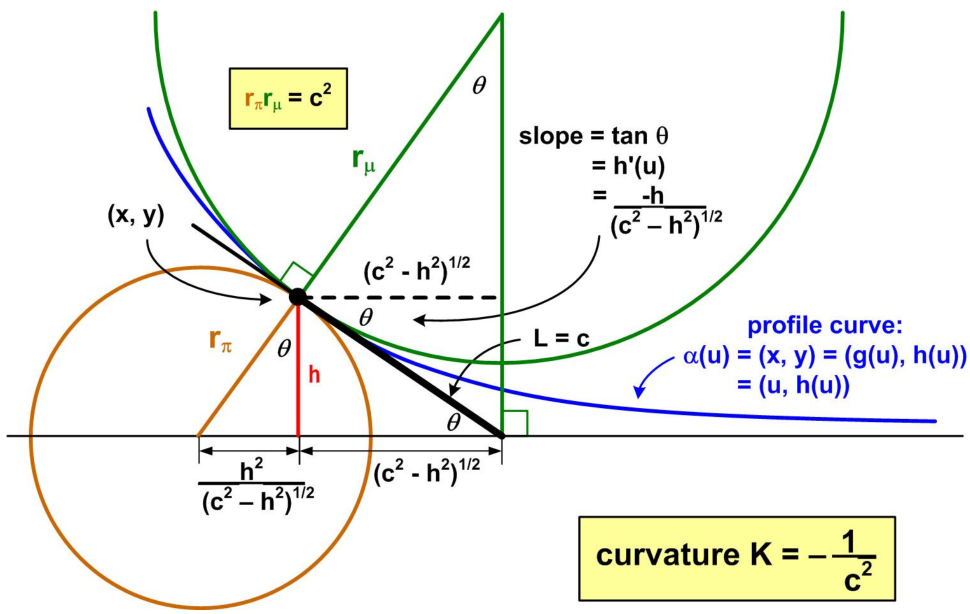

Figure 12 Tractroid, Bugle, Beltrami Surface (tractrix profile curve) with Gaussian curvature K and radii of curvature r_π, r_μ

At last we have a constant negative Gaussian curvature. As with the other cases, the radius of curvature of the parallel $r_\pi = 1/k_\pi = ch/(c^2 - h^2)^{1/2}$ (ignoring signs) agrees with the argument given above illustrated in Figure 4 where r_π is shown to be the hypotenuse of the triangle with legs h and $h^2/(c^2 - h^2)^{1/2}$ ($h^2/(c^2 - h^2)^{1/2} = h \tan \theta$).

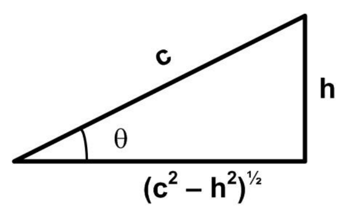

Figure 13 Trigonometric Relations

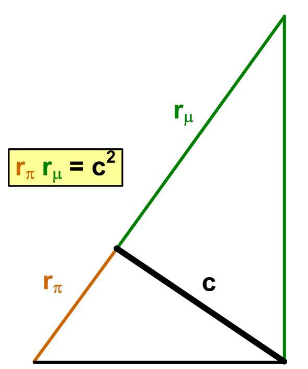

Figure 14 Geometric Mean

We showed how in the previous examples the length L varied as the point moved along the profile curve, whereas for the tractrix it is constant. The other term we have been looking at is the x-distance of the tangent point from the center of the osculating circle with radius of curvature r_μ. In the case of the tractrix, the center of the circle is directly over the point where the tangent line intersects the x-axis. This produces a right triangle exhibiting the geometric mean property that relates the constant curvature to the constant length of L = c (Figure 14). This is a very nice graphical way of showing how intimately the tractrix constant length property is tied in with the constant curvature property.

We have not given an expression for the x and y values of the actual profile curve. It turns out a fairly simple parametric version is given by ([10]):

$$x = g(u) = u - \tanh(u) \tag{5}$$

$$y = h(u) = \operatorname{sech}(u)$$

where u > 0.

Exponential Representation

This is all very fine, but where is the exponential that Khovanova and others said produced the constant negative curvature? If we parameterize the tractrix by u = arclength, then the equations become ([10]):

$$x = g(u) = \ln\left(e^{u} + \sqrt{e^{2u} - 1}\right) - e^{-u}\sqrt{e^{2u} - 1} \tag{6}$$

$$y = h(u) = e^{-u}$$

for u > 0. Finally, an explicit exponential for the y-values.

But parameterizations are tricky. The x-value parameterization must also be taken into account. That is, we might think a curve given by $y = h(u) = u^2$ represents a parabola, but if $x = g(u) = u^2$ as well, then the curve is actually y = x or a straight line.

So what is going on? How is the tractrix profile curve parameterized by arclength related to the exponential profile curve of the previous example? Hopefully Figure 15 provides the answer.

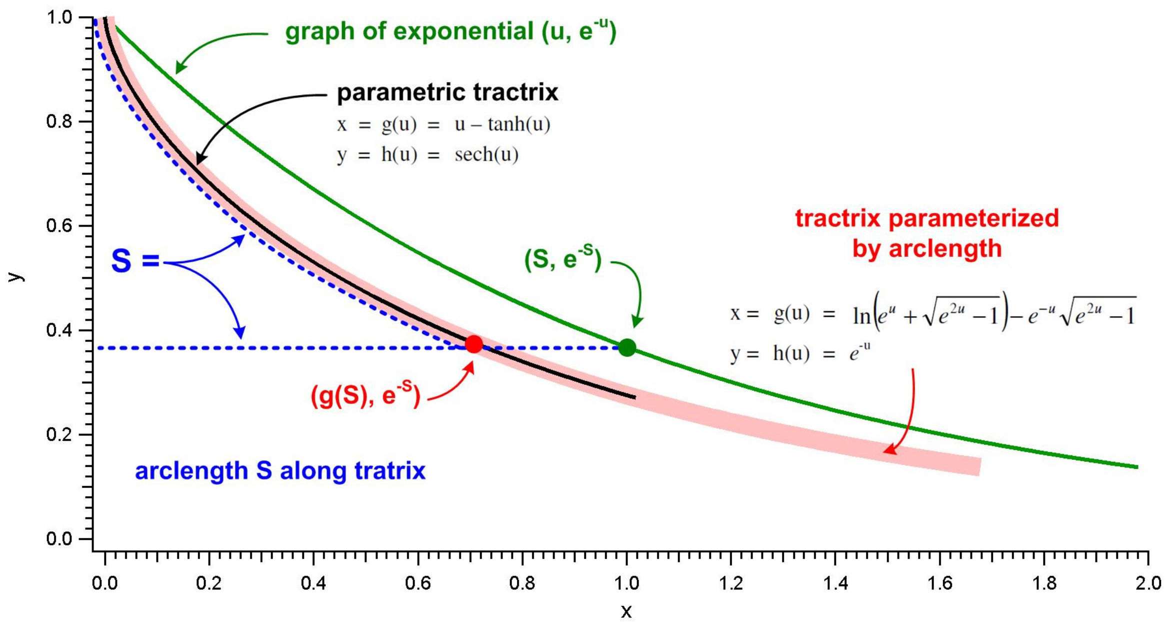

Figure 15 Tractrix vs. Exponential Profile Curves

I used the Igor analytic tool to plot three profile curves: the (green) exponential graph (u, e^{-u}), the (black) tractrix using the parametric equations (5), and the (red) tractrix using the arclength parameterized equations (6). I fattened the red curve to show how it coincided with the other parameterization for the tractrix. S measures the arclength along the tractrix as well as the equal horizontal distance to the exponential graph. The idea is that the y-value e^{-S} is the same in both cases.

My Error

So what was Tanya Khovanova really saying in her crochet posting and where did I go wrong? Another picture should make it clear. I was effectively parameterizing the crochet loops along the x-axis, that is, I was using equal horizontal spacing for the loops (Figure 16). Khovanova was effectively using equal arclength spacing (Figure 17). If the figures are reversed to conform to the surfaces of revolution we have been considering, then it is easy to see the same compression along the x-axis for the arclength parameterization in Figure 17 is likewise shown in Figure 15.

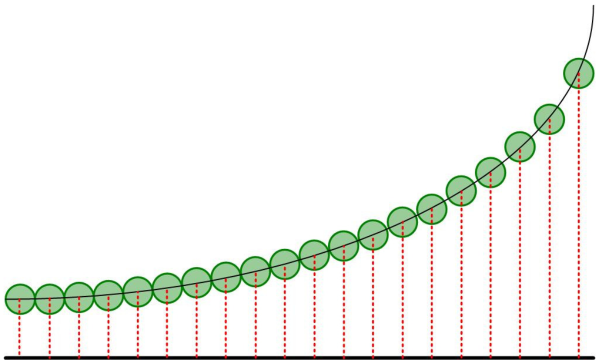

Figure 16 Crochet Loops Equal Horizontal Spacing

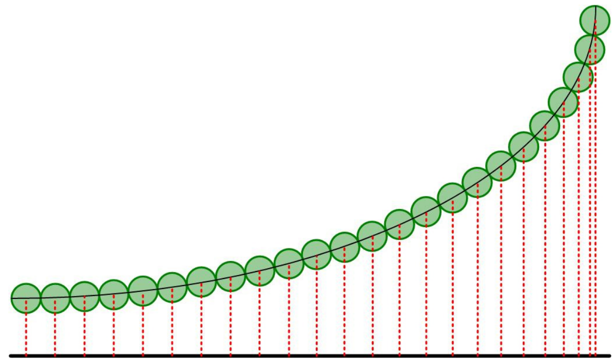

Figure 17 Crochet Loops Equal Arclength Spacing

So Khovanova was really showing dC/ds = 2C where C is the circumference of the crochet loop and s is the arclength measure along the surface. Since $C = 2\pi r$, we have dr/ds = 2r rather than dr/dx = 2r as I was indicating. Thus the mystery is resolved. Details *really* matter in mathematics, much to my consternation at times when I try to simplify things.

Addendum – Intrinsic Geometry

I mentioned before that the Gaussian curvature is a quantity intrinsic to the surface. It is what someone living in the surface would measure. The statements about the crocheted surface that rely on arclength are of a similar nature since arclength is what someone would measure on the crochet surface.

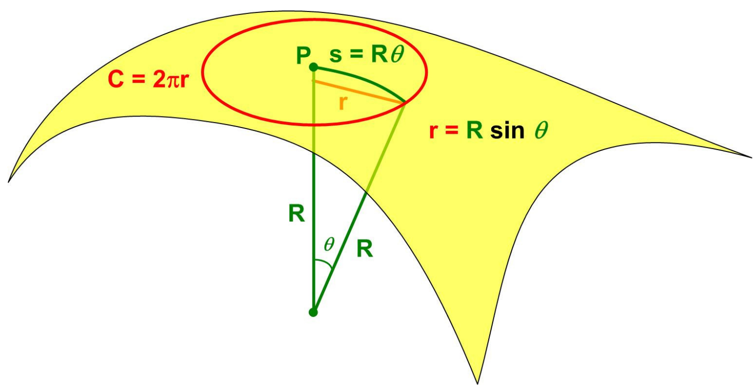

Figure 18 Surface with Constant Positive Curvature

We can see a similar result for a spherical surface with constant positive curvature (Figure 18). The "radius" of a circle in the surface centered at P is given by the arclength $s = R\theta$ of the great circle from the center to the circumference of the circle. The circumference C is parameterized by s via the relations $C = 2\pi r = 2\pi R \sin\theta = 2\pi R \sin s/R$. Therefore the rate of growth of the circumference C with respect to "radius" s is

$$dC/ds = 2\pi \cos s/R \leq 2\pi$$

For a flat plane, the radius of a circle is the same as the arclength, so $dC/ds = dC/dr = 2\pi$, a constant. Hence we see that a constant positive Gaussian curvature indicates a slower growth of the circumference with respect to the "intrinsic" radius than in a flat plane. Similarly, as we saw with the tractroid, for a constant negative Gaussian curvature we have $dC/ds = 2\pi\, dr/ds = 2\pi\, k\, e^s \geq 2\pi$ for $k > 1$. So the growth of the circumference in this case is greater than 2π.

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