Turnpike Driving

30 January 2019

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This turns out to be a fairly challenging driving problem from Longley-Cook ([1], #86 p.55).

"Mileage on the Thru-State Turnpike is measured from the Eastern terminal. Driver A enters the turnpike at the Centerville entrance, which is at the 65-mile marker, and drives east. After he has traveled 5 miles and is at the 60-mile marker, he overtakes a man operating a white-line painting machine who is traveling east at 5 miles per hour. At the 35-mile marker he passes his friend B, whose distinctive car he happens to spot, driving west. The time he notes is 12:20 p.m. At the 25-mile marker he passes a grass cutter traveling west at 10 miles per hour. A later learns that B overtook the grass cutter at the 21-mile marker and passed the white-line painter at the 56-mile marker. Assuming A, B, the painter and the grass cutter all travel at constant speeds, at what time did A enter the turnpike?

Solution

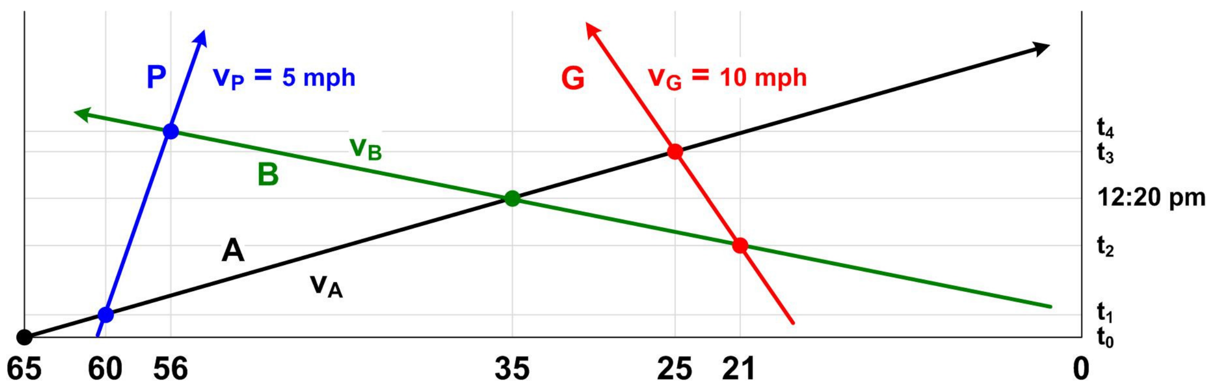

Figure 1 Problem Statement

Figure 1 shows the problem setup with P representing the white-line painter and G the grass cutter. We make the time and distance relationships explicit in Figure 2 where we have added the times for the line painter and the grass cutter to go 4 miles. From Figure 2 we can also get equations for the times drivers A and B take to go between the points of meeting the other drivers. We get

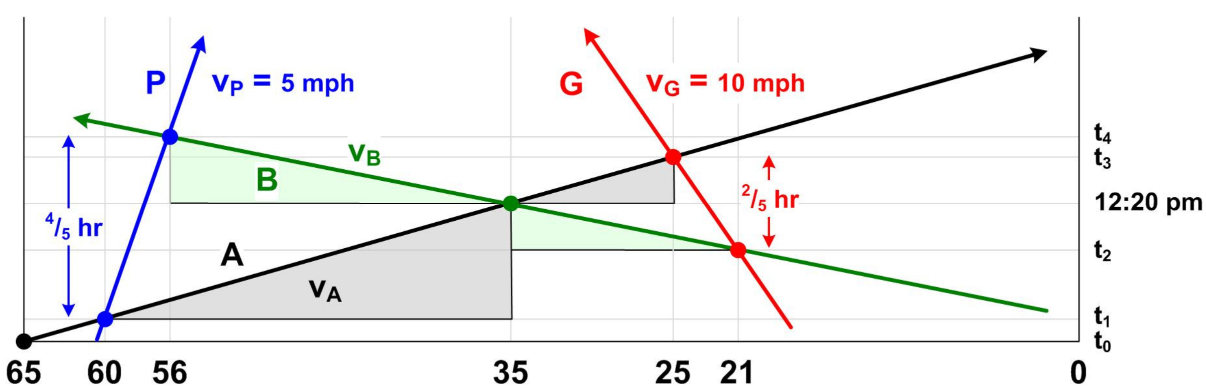

Figure 2 Problem Solution

$$\frac{4mi}{10mph} = \frac{10mi}{v_A} + \frac{14mi}{v_B}$$

$$\frac{4mi}{5mph} = \frac{25mi}{v_A} + \frac{21mi}{v_B}$$

If we let $x = 1/v_A$ and $y = 1/v_B$, then we have the following simultaneous linear equations in two unknowns.

$$10x + 14y = 2/5$$

$$25x + 21y = 4/5$$

or

$$50x + 70y = 2$$

$$125x + 105y = 4$$

We solve the equations via Cramer's Rule, that is, $x = D_1/D_0$ and $y = D_2/D_0$ where

$$D_0 = \begin{vmatrix} 50 & 70 \\ 125 & 105 \end{vmatrix} = 25 \cdot 35 \begin{vmatrix} 2 & 2 \\ 5 & 3 \end{vmatrix} = -4 \cdot 25 \cdot 35$$

$$D_1 = \begin{vmatrix} 2 & 70 \\ 4 & 105 \end{vmatrix} = 2 \cdot 35 \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -2 \cdot 35$$

$$D_2 = \begin{vmatrix} 50 & 2 \\ 125 & 4 \end{vmatrix} = 2 \cdot 25 \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -2 \cdot 25$$

Therefore

$$x = D_1/D_0 = (-2\cdot35)/(-4\cdot25\cdot35) = 1/50 \;\Rightarrow\; v_A = 50$$

and

$$y = D_2/D_0 = (-2\cdot25)/(-4\cdot25\cdot35) = 1/70 \;\Rightarrow\; v_B = 70.$$

Thus

$$12{:}20 - t_0 = 30/\,v_A = 30/50 = 3/5$$

$$t_0 = 12\,{}^1\!/_3 - 3/5 = 176/15 = 11\,{}^{11}\!/_{15} = 11\,{}^{44}\!/_{60}$$

So the time driver A entered the thru-way is t_0 = 11:44 am.

References

[1] Longley-Cook, L. H., *Work This One Out*, Ernest Benn Ltd., 1960, Fawcett Publications, Crest Reprint, 1962.