Three Coffin Problems

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I had vaguely remembered about these problems, but a video by GoldPlatedGoof (Nakul Dawra) reminded me with three excellent examples.([1]) Here is a description of the Coffin Problems from Tanya Khovanova, one of the students involved ([2]):

> The Mathematics Department of Moscow State University, the most prestigious mathematics school in Russia, was at that time [1975] actively trying to keep Jewish students (and other "undesirables") from enrolling in the department. One of the methods they used for doing this was to give the unwanted students a different set of problems on their oral exam. I was told that these problems were carefully designed to have elementary solutions (so that the Department could avoid scandals) that were nearly impossible to find. Any student who failed to answer could easily be rejected, so this system was an effective method of controlling admissions. These kinds of math problems were informally referred to as "Jewish" problems or "coffins". "Coffins" is the literal translation from Russian; they have also been called "killer" problems in English.

Of the three problems selected by Dawra I was able to solve the first two (more or less along the lines given by Dawra), but after playing with it for some time, I could not solve the third problem and had to look at Dawra's solution. The third problem seemed to be a premier example of a difficult-to-think-of-solution that was nevertheless exceedingly simple—*once you saw it*.

More extensive information and history of the Coffin Problems can be found at Tanya Khovanova's website entry ([3]), including Coffin problems from other mathematicians and numerous links to other material.

References

[1] GoldPlatedGoof (Nakul Dawra), "Simple Math Problems To Fool The Best" (https://www.youtube.com/watch?v=Db3CwFh7qng) 25 Aug 2017

[2] Khovanova, Tanya and Alexey Radul, "Jewish Problems," October 18, 2011 (arXiv:1110.1556v2 (https://arxiv.org/abs/1110.1556v2) [math.HO])

[3] Khovanova, Tanya, "Coffins" August 2008 (http://www.tanyakhovanova.com/coffins.html).

First, I will give the problems and then the solutions.

Problem 1

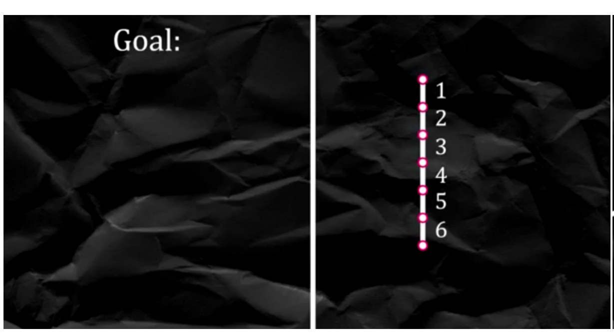

Figure 1

Figure 2

You are given two vertical, parallel lines. The goal is to divide the smaller line into six equal-length intervals (Figure 1) using only straight lines and their intersections (Figure 2).

Problem 2

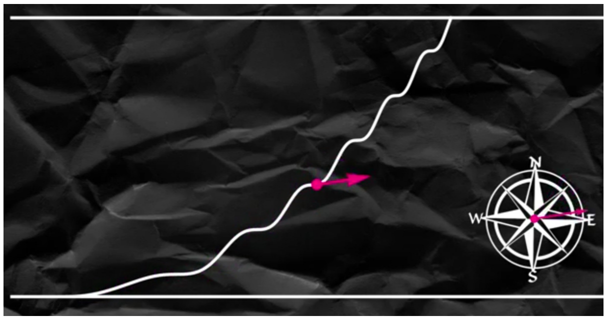

Figure 3

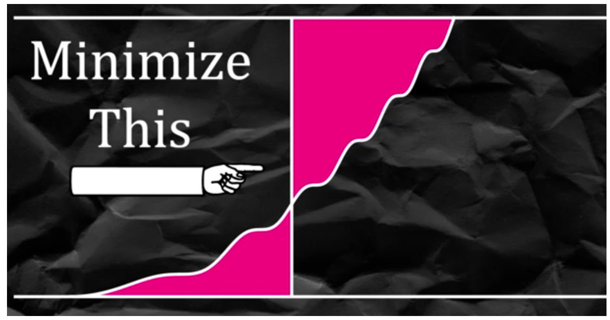

Figure 4

In this problem you are given two parallel horizontal lines and an arbitrary monotonically increasing curve between them (Figure 3). Dawra characterizes this curve by saying a tangent arrow will always point between East and North on the compass. A vertical line is then drawn between the two parallel lines cutting the curve and forming two areas, one to the left below the curve and one to the right above the curve (Figure 4). This vertical line can move back and forth, changing the sizes of the two areas. The goal is to find (for any arbitrary such curve) the location of the vertical line that minimizes the sum of the two areas.

I really like this problem. There is something quite elegant about it.

Problem 3

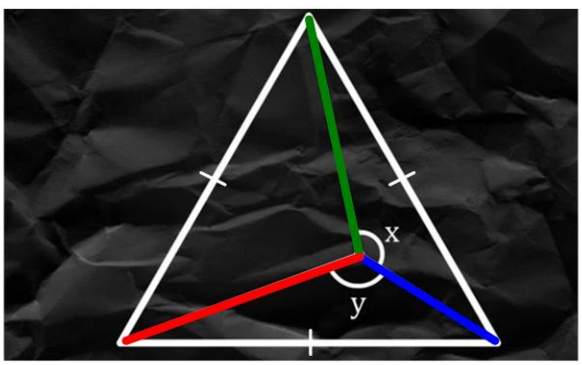

Figure 5

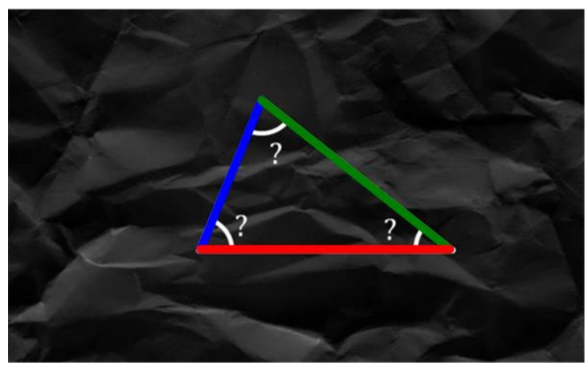

Figure 6

You are given an equilateral triangle, an arbitrary point inside, and three lines joining the point to the three vertices (Figure 5). If the three lines are rearranged into a triangle, what are the interior angles of this new triangle using the angles x and y in the original figure? (Figure 6).

I had actually seen this problem before on Presh Talwalkar's website, *Mind Your Decisions*, and could not solve it then. On the face of it, there just seems to be no obvious connection between the two figures. But once you see the solution, it is practically trivial.

Solution to Problem 1

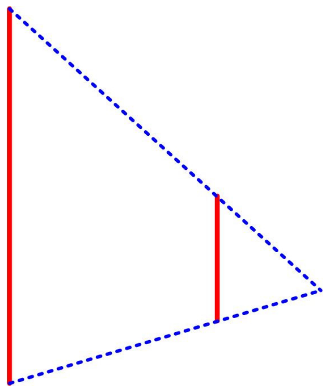

Figure 7 Initial Step

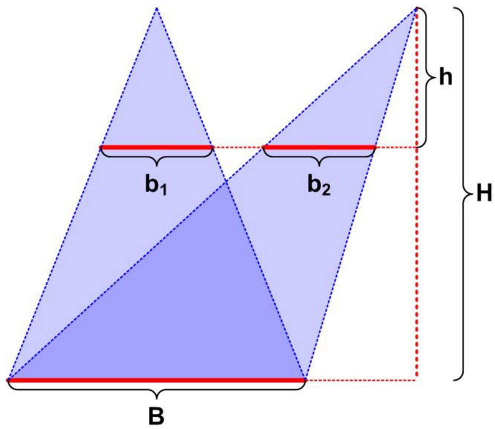

Figure 8 Shear Invariance

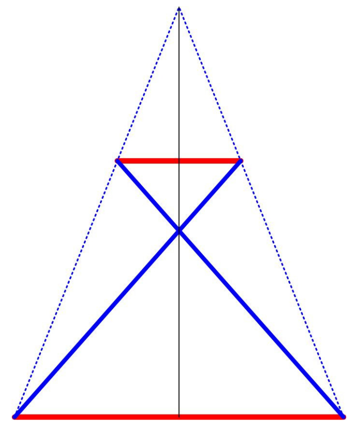

Figure 9 Initial Attempt

The only obvious points to join initially with straight lines are the end points of the line segments making a triangle (Figure 7). Also, parallel lines in triangles can be associated with proportional sides and so may provide a way to solve the problem. The easiest triangles to work with are right triangles and isosceles triangles, so I wanted to shift the problem to such figures. But I had to make sure this would not change the problem.

Figure 8 shows a skew triangle can be "sheared" to an isosceles triangle by moving its top vertex parallel to its base. Similar triangles are preserved. That is, in the skew triangle we have (via similar triangles) $b_2 / B = h / H$. But since the altitudes are preserved via the shearing, we also have $b_1 / B = h / H$, so that $b_1 = b_2$. We will call this preservation of lengths under shearing, shear invariance.

Figure 9 shows my initial attempt at a solution by drawing the only other obvious lines connecting intersections. The blue lines produced a new intersection, which when joined by a line from the vertex to the base, "looked" like a perpendicular bisector. But I could not see easily how to prove it. I kept slipping into assuming what I wanted to prove, that is, I was beguiled by the picture and had difficulty establishing rigorous statements.

I thought this intersection property was very interesting and began drawing a succession of crossing lines. First I drew a line parallel to the base through the original intersection of the blue lines. Then I connected the base vertices to the new line endpoints to form more intersections. I kept

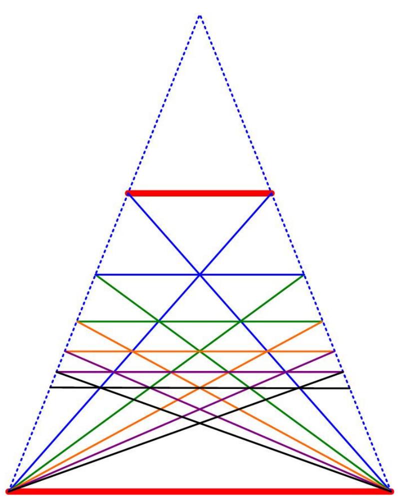

Figure 10 Crossing Lines

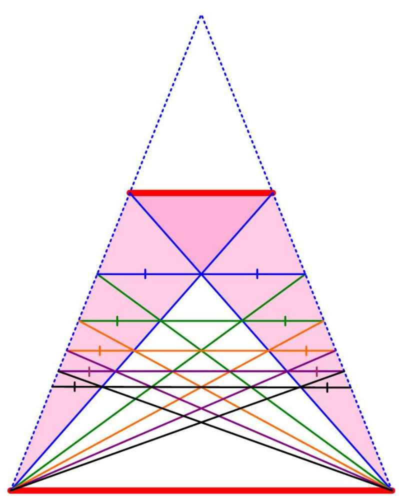

Figure 11 1st Equal Segments

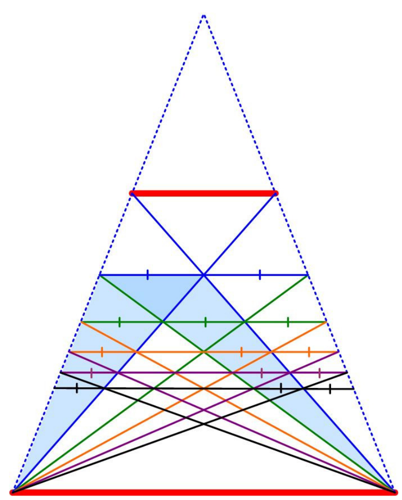

Figure 12 2nd Equal Segments

doing this with more horizontal lines joined by intersecting lines from the base vertices (Figure 10). The result was fascinating. It "looked like" each horizontal segment was cut into equal length segments in consecutive numbers: 2, 3, 4, 5, 6. How to prove it? Aha, by tipping the shear invariance diagram upside down I could show each horizontal line segment was cut into equal subsegments (Figure 11, Figure 12, and Figure 13).

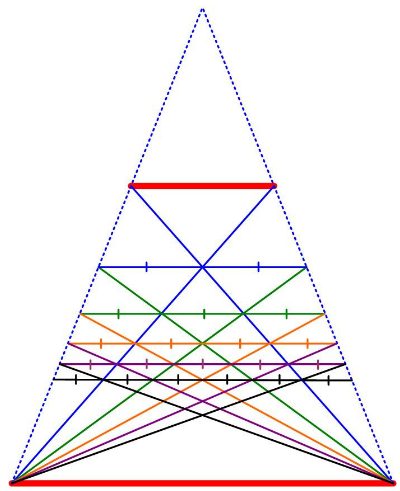

Figure 13 Final Equal Segments

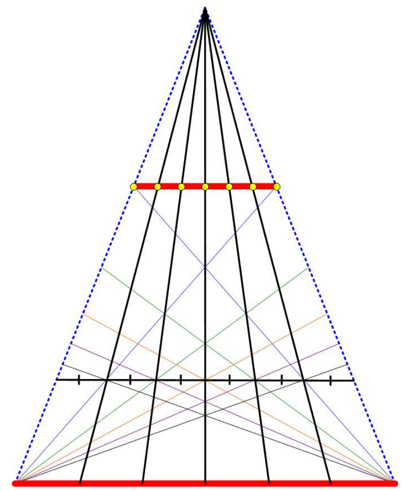

Figure 14 Final Solution

Joining the intersections in the last horizontal segment with lines from the vertex of the triangle cut the smaller line into 6 equal segments as required for the problem solution (Figure 14).

Nakul Dawra's Variation

Dawra did virtually the same thing as I did. But he stopped at my initial attempt (Figure 9) using a skew triangle and just asserted *without proof* that the line from the vertex through the crossing lines bisected the base. He then continued to bisect the base segments into equal subsegments getting 2, 4, and $2^3 = 8$ equal subsegments. He then used the first 6 subsegments and lines from the vertex of the triangle to cut the smaller line into 6 equal segments.

So I felt my approach at least was fully proved, and it also produced a very fascinating pattern of equal segments from line intersections down the triangle towards the base. In addition there was the interesting facet that this pattern was produced no matter what the starting line segment was.

Solution to Problem 2

Figure 15 shows the solution. Without loss of generality, I assumed the parallel lines were 1 unit apart and the horizontal extent of the curve was also 1 unit. My reasoning, which agreed with Dawra's in this case, went as follows. As the vertical line (from x through f(x)) moves to the right in the figure, the upper blue area will decrease and the lower red area increase. To the left of the point x_0 the addition from the red area is less than the decrease from the blue area, so the total area is shrinking. To the right of the point x_0 the addition from the red area is greater than the decrease from the blue, so the total area is increasing. Therefore, it is right where the increase in the red area exactly balances the decrease in the blue area that the total area is minimal. This happens to occur when the vertical height of the curve f(x) is exactly ½.

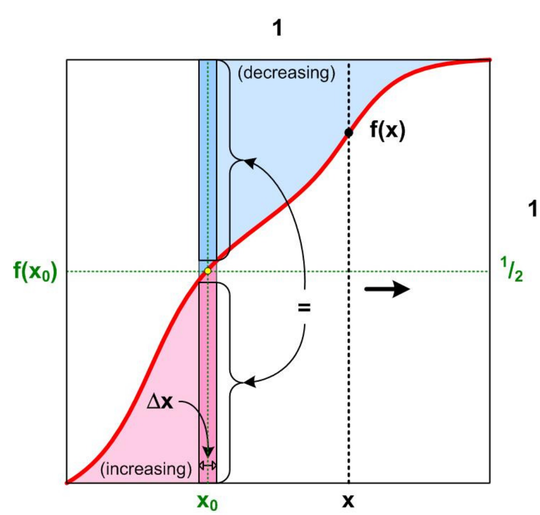

Figure 15 Minimum Area Solution

Therefore to find where the vertical line should be drawn for the minimal total area, first find where the horizontal line at ½ crosses the curve and draw the vertical through that point.

Calculus Solution

I am not sure whether the Russian students would know calculus for this exam, but there is an easy solution using it. Namely, the total area is given by the integrals

$$A(x) = \int_0^x f(t)dt + \int_x^1 (1 - f(t))dt = \int_0^x f(t)dt - \int_1^x (1 - f(t))dt$$

Therefore, the minimum will occur for the x_0 where the derivative vanishes, that is,

$$dA/dx = A'(x_0) = f(x_0) - (1 - f(x_0)) = 0$$

or

$$f(x_0) = 1/2$$

just as before.

Solution to Problem 3

Just rotate a copy of the original equilateral triangle 60° counterclockwise and place it adjacent to the original (Figure 16). Then voila! All the interior angles of the new (shaded) triangle are the same as those centered on the original lines minus 60°.

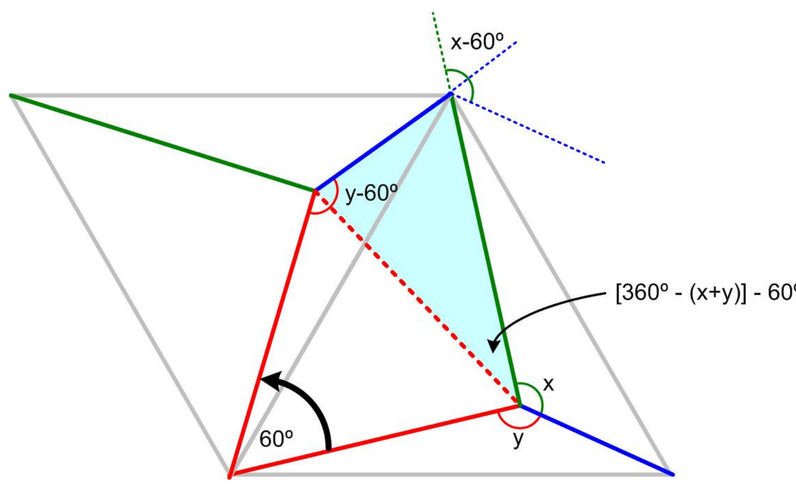

Figure 16 Problem 3 Solution

One detail: the figure shows the addition of a red dashed line closing off an equilateral (red) triangle. When the (solid) red line is rotated 60°, the two lines become the equal sides of a triangle with vertex 60°. That means joining their free ends makes an equilateral triangle, whose new side forms the small triangle in the problem. Amazing!

Even though I was able to solve the first two problems, it still took me considerable time—several hours to think up the approaches and to work out the details. Khovanova said the exam was oral, so I am sure under such pressure I could never have found the solutions in time.