Right Triangle with Roots

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This is an interesting problem from the United Kingdom Mathematics Trust (UKMT) Senior Math Challenge of 2008.

The length of the hypotenuse of a particular right-angled triangle is given by

$$\sqrt{1+3+5+7+...+23+25}\,.$$

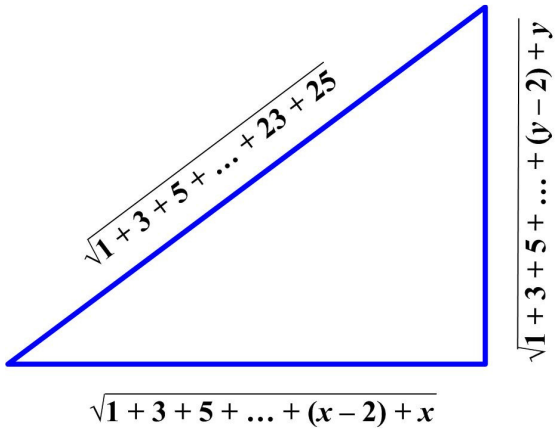
The lengths of the other two sides are given by

$$\sqrt{1+3+5+7+...+(x-2)+x}$$

and

$$\sqrt{1+3+5+7+...+(y-2)+y}$$

where x and y are positive integers. What is the value of $x + y$?



Solution.

This relies on the formula for the arithmetic sum of odd integers, which can be derived from the arithmetic sum of all integers as follows.

$$1 + 3 + 5 + \ldots + (2n + 1) = \sum_{k=0}^{k=n} 2k + 1 = 2\sum_{k=0}^{k=n} k + (n+1) = 2\frac{n(n+1)}{2} + (n+1) = (n+1)^2 \quad (1)$$

From the Pythagorean theorem we have

$$1 + 3 + 5 + \ldots + 23 + 25 = (1 + 3 + 5 + \ldots + (x - 2) + x) + (1 + 3 + 5 + \ldots + (y - 2) + y)$$

or from equation (1) with $25 = 2\cdot12 + 1$, $x = 2m + 1$, and $y = 2n + 1$,

$$(13)^2 = (m + 1)^2 + (n + 1)^2$$

The only Pythagorean triple with 13 as the hypotenuse is a 5-12-13 triangle. So $m = 4$ and $n = 11$, which implies that $x = 2m + 1 = 9$ and $y = 2n + 1 = 23$, so that $x + y = 32$.