Pool Party

(13 February 2018)

Jim Stevenson

Futility Closet offers another interesting puzzle:

(https://www.futilitycloset.com/2012/01/08/pool-party/, retrieved 2/12/2018)

A billiard ball is resting on a table that measures 10 feet by 5 feet. A player hits it with no "English" and it strikes four different cushions and returns to its starting point. University of Alberta mathematician Murray Klamkin asks: How far did it travel? (8 January 2012)

Solution

After solving the problem myself, I verified that *Futility Closet* provides an answer, but without real justification. So I thought I would write up my solution.

Billiard table problems generally rely on the mirrored image of the trajectories of the billiard balls that bounce off the sides of the table in perfect reflections. That is, the angle of incidence of the ball equals the angle of reflection. This idea is shown in Figure 1.

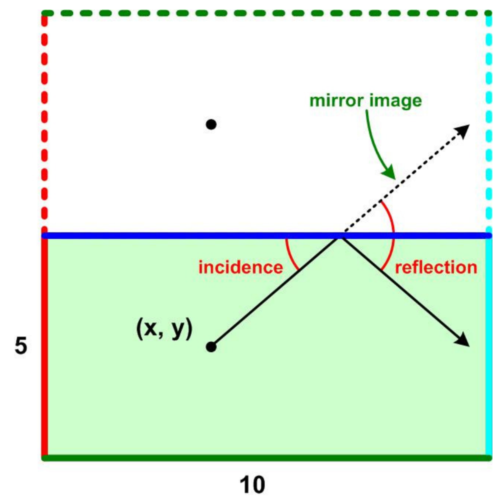

Figure 1 Mirror image of reflected ball

Since in the problem the ball has to bounce off of each cushion, I have colored the cushions differently.

Figure 2 shows a number of reflected images of the billiard table and possible paths for the billiard ball. Since the ball has to return to its original position, that means there are only a discrete number of paths allowed, one for each start position in the mirrored image of the original billiard table.

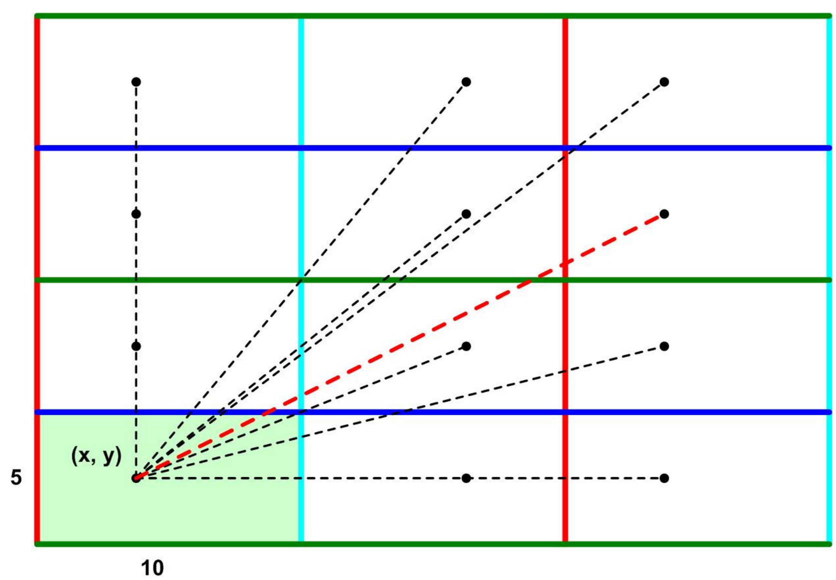

Figure 2 Discrete Solutions

Notice that for the ball to arrive at the positions on the top row, it has had to bounce off the top (blue) cushion twice. But the problem says that the balls must bounce of each cushion *exactly* once. In terms of the reflected images this means the trajectory of the ball must cross each colored line once and only once. That means it must end up in the second row above the original table and the second column to the right, as shown by the red dashed line. (We ignore the cases where the ball hits the corners of the billiard table, since then they would not bounce off each cushion.)

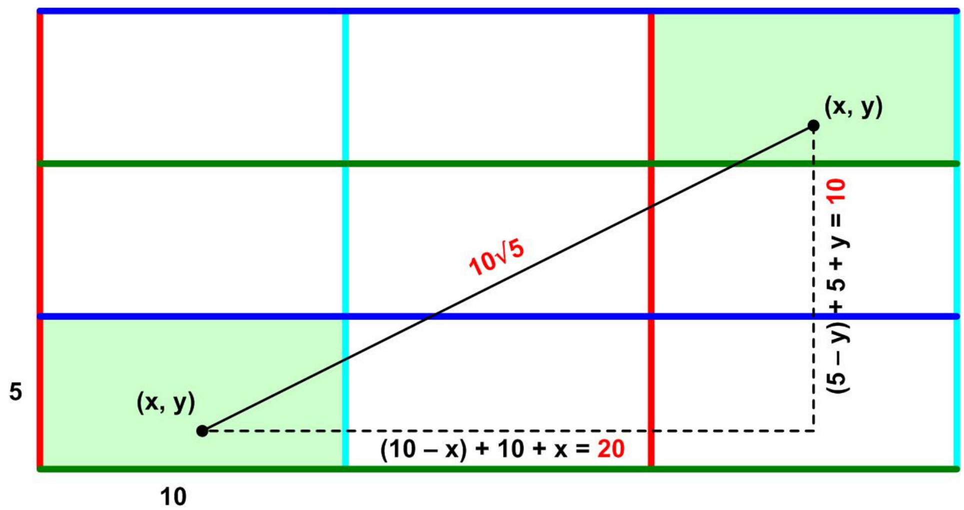

Figure 3   Solution

Figure 3 shows the final solution to the problem. We can measure the distance between the start and end position via the Pythagorean theorem using the horizontal and vertical displacements. Namely, the horizontal displacement is the sum of the displacements in the first, second and third columns:

$$(10 - x) + 10 + x = 20$$

and the vertical displacement is:

$$(5 - y) + 5 + y = 10$$

Therefore the diagonal distance is

$$\sqrt{20^2 + 10^2} = 10\sqrt{5} \approx 22.36$$

Notice that because the x and y terms cancel, this distance is independent of the start position (x, y). Also notice that since the horizontal and vertical displacements are also independent of x and y, the trajectories for all starting points (x, y) and heading upward and to the right are parallel.

One final note. Our picture showed the case where the initial direction of the ball was upward and to the right. It is easy to see any other direction would yield a mirror image (pardon the quasi-pun) of our first solution. But this would not change the distance measurement.