Polygon Altitude Problems II

(1 September 2018, rev 3 October 2018)

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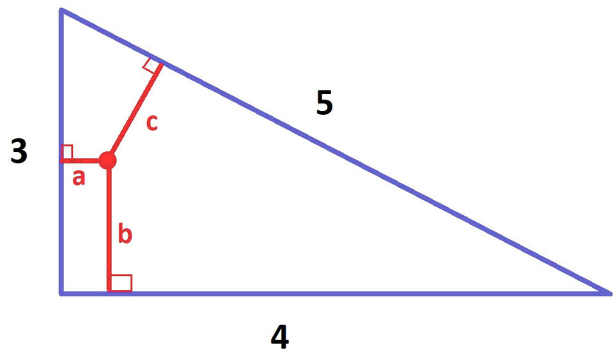
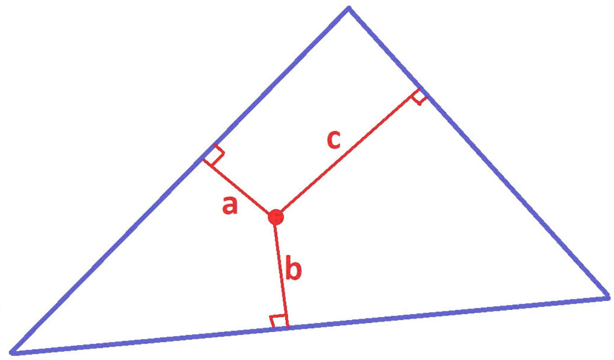
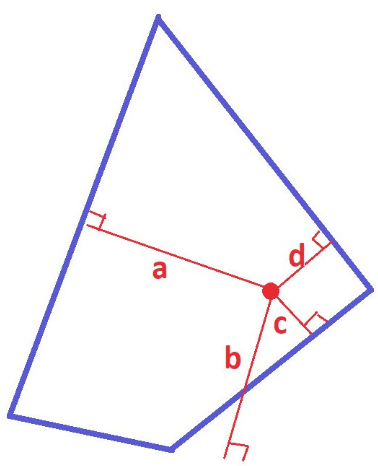
James Tanton has provided further elaborations on the polygons and the sum of perpendicular distances from interior points.



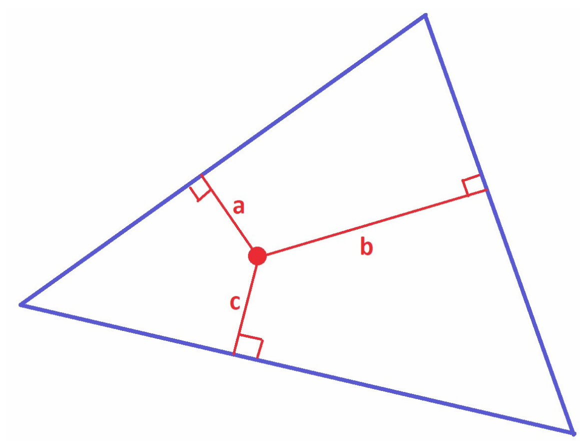
Problem 1.[1] Show that for each point inside a 3-4-5 right triangle with distances a, b, c shown the value of 3a + 4b + 5c is 12.



Problem 2.[2] (Having fun pushing Viviani Thm to the max!) A triangle has the property that for each point inside the triangle the sum of the three distances shown, a+b+c, has the same fixed value. Must the triangle be equilateral? (Bonus: Find a triangle for which a+b+2c is a fixed value.)



Problem 3.[3] A convex quadrilateral has the property that for each point inside the figure the sum of its distances from each side (maybe extend a side length) is the same fixed value. Must the quadrilateral be a parallelogram?



Problem 4.[4] A triangle has the property that for three fixed values r, s, t the sum ra + sb + tc is the same for all points inside the triangle where a, b, c are the distances shown. What can you say about the shape of the triangle?

[1] https://twitter.com/jamestanton/status/1030431045496320000, 17 August 2018
[2] https://twitter.com/jamestanton/status/1030655003881623553, 17 August 2018
[3] https://twitter.com/jamestanton/status/1030993833343868928, 18 August 2018
[4] https://twitter.com/jamestanton/status/1032180719185842176, 22 August 2018

Problem 5.[5] A convex polygon has the property that for each point inside the figure the sum of its distances to each side (extending side lengths if necessary) is the same fixed value. Must the polygon possess rotational symmetry of some degree? (eg This is a true property of all reg polys)

Solution to Problem 1

As in the previous set of problems of this type ([1]), we subdivide the right triangle into three non-overlapping triangles with bases each of the sides of the right triangle and top vertex the arbitrary internal point (Figure 1). Then the area of the entire triangle equals the sum of the areas of the internal triangles.

$$A = ½\ 4{\cdot}3 = ½\ 3a + ½\ 4b + ½\ 5c$$

$$\therefore\ \ 12 = 3a + 4b + 5c.$$

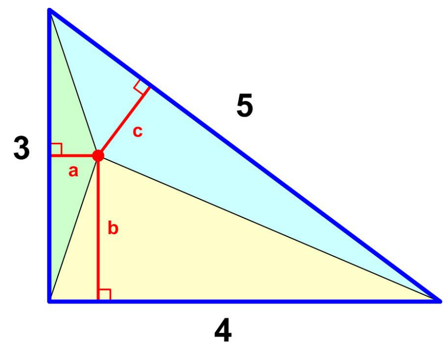


Figure 1

Solution to Problem 2

Again we proceed as in the previous problems by adding the unit vectors corresponding to the three perpendiculars. We start with the point P in the center (center of an inscribed circle, which means all three perpendiculars are equal) and then move it to any point P' in the interior of the triangle (Figure 2). The translation vector v is drawn from P to P' and its projections onto each of the unit vectors provide the change in lengths for the corresponding perpendiculars. So the sum of these projections gives the net change in the lengths of the perpendiculars, which equals zero by the problem statement that a + b + c is constant, that is,

$$\mathbf{v}{\cdot}\mathbf{u_a} + \mathbf{v}{\cdot}\mathbf{u_b} + \mathbf{v}{\cdot}\mathbf{u_c} = \mathbf{v}{\cdot}(\mathbf{u_a} + \mathbf{u_b} + \mathbf{u_c}) = \mathbf{v}{\cdot}\mathbf{u} = 0 \qquad (1)$$

Since the unit vectors represent a 90° rotation of the edges of the triangle, their head-to-tail sum tries to mimic the original triangle only rotated 90°. But since their lengths are all equal, if the figure is closed, it must represent a regular polygon—in this case an equilateral triangle. Since the original triangle is not equilateral, then the unit vector sum must result in a non-zero vector **u** (Figure 3).

Now since P' is any point in the interior of the triangle, we can pick a suitable **v** as follows. Let

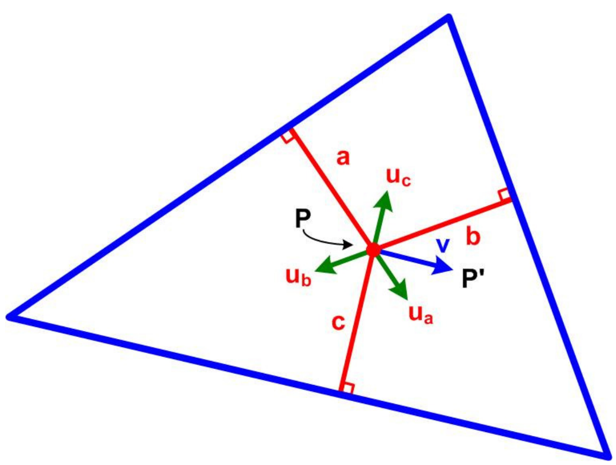


Figure 2 Unit vectors u_a, u_b, u_c and translation vector v for irregular (scalene) triangle

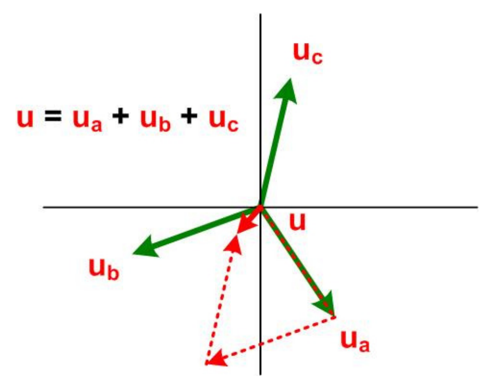


Figure 3 Non-zero sum of unit vectors

[5] https://twitter.com/jamestanton/status/1031462532391870465, 20 August 2018

$k = ½ \min(a, |\mathbf{u}|)$ where $a = b = c$ for the initial, center position P, and $|\mathbf{u}|$ is the length of the vector $\mathbf{u}$. Then let $\mathbf{v} = k\,\mathbf{u}/|\mathbf{u}|$. This will represent a move from P to P' that is still inside the triangle. Then equation (1) yields

$$0 = \mathbf{v}\cdot\mathbf{u} = k\,\mathbf{u}\cdot\mathbf{u}/|\mathbf{u}| = k\,|\mathbf{u}|$$

But that implies $|\mathbf{u}| = 0$, contrary to our assumption. Therefore, in order for the sum of the perpendiculars to remain constant, the triangle must be equilateral.

Triangle where a+b+2c is a fixed value.

Using the area idea in the solution of Problem 1, it looks like there is no triangle satisfying this. For suppose there is, then $a + b + 2c = k_1$, a constant and $aS_1 + bS_2 + cS_3 = k_2$, a constant, where S_1, S_2, and S_3 are the respective sides of the triangle and k_2 represents twice the area of the triangle. Then

$$aS_1\, k_1/k_2 + bS_2\, k_1/k_2 + cS_3\, k_1/k_2 = k_1$$

$$a + b + 2c = k_1$$

implies $$a(S_1\, k_1/k_2 - 1) + b(S_2\, k_1/k_2 - 1) + c(S_2\, k_1/k_2 - 2) = 0$$

Therefore, $b = c = 0 \Rightarrow S_1 = k_2/k_1$, and $a = c = 0 \Rightarrow S_2 = k_2/k_1$, so that $S_1 = S_2$, and $a = b = 0 \Rightarrow S_3 = 2k_2/k_1 = 2\,S_1 = 2\,S_2$. But this means $S_3 = S_1 + S_2$, which contradicts the fact that the sum of any two sides of a triangle must be *larger* than the third side. So there is no triangle satisfying $a + b + 2c$ is a constant for all perpendiculars a, b, c from a random point P' inside the triangle.

Solution to Problem 3

We proceed as in the solution for Problem 2. In order that the sum of the 4 perpendiculars be constant, the corresponding head-to-tail sum of unit vectors must vanish and that implies a 4-sided figure with all sides equal. This can be a square or rhombus. But we don't need all 4 sides to be equal. As with our solution to Problem 4 in *Polygon Altitude Problems I*, it suffices if there are subsets of rotationally symmetric sides. In this case of a quadralateral, such figures imply that opposite sides of the original figure at least must be parallel, which means the most general polygon would be a parallelogram.

Solution to Problem 4

We proceed as in the solution for Problem 2 regarding the triangle with constant $a + b + 2c$ value. We have $ar + bs + ct = k_1$, a constant for fixed r, s, and t and variable a, b, and c (corresponding to points in the triangle). Again we have $aS_1 + bS_2 + cS_3 = k_2$, a constant, where S_1, S_2, and S_3 are the respective sides of the triangle and k_2 represents twice the area of the triangle. Therefore,

$$aS_1\, k_1/k_2 + bS_2\, k_1/k_2 + cS_3\, k_1/k_2 = k_1$$

$$ar + bs + ct = k_1$$

implies $$a(S_1\, k_1/k_2 - r) + b(S_2\, k_1/k_2 - s) + c(S_2\, k_1/k_2 - t) = 0$$

Therefore, $b = c = 0 \Rightarrow S_1 = r\,k_2/k_1$, and $a = c = 0 \Rightarrow S_2 = s\,k_2/k_1$, and $a = b = 0 \Rightarrow S_3 = t\,k_2/k_1$. Hence the sides of the triangle are proportional to r, s, and t, respectively, that is,

$$S_1 / r = S_2 / s = S_3 / t = k_2/k_1$$

Solution to Problem 5

To show the polygon does not need to have rotational symmetry, we only need is an example of one such polygon with constant sum of perpendiculars which has no rotational symmetries. Figure 4 provides such an example with an irregular (equilateral) pentagon with no rotation symmetries.

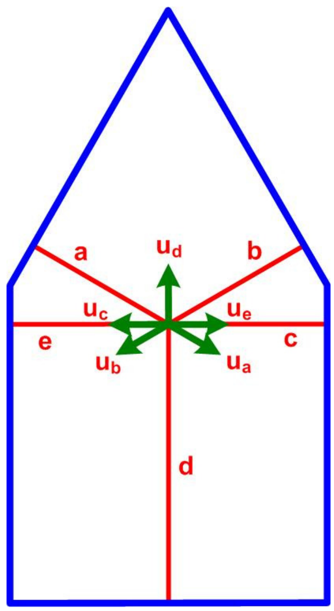


Figure 4 Irregular (equilateral) pentagon

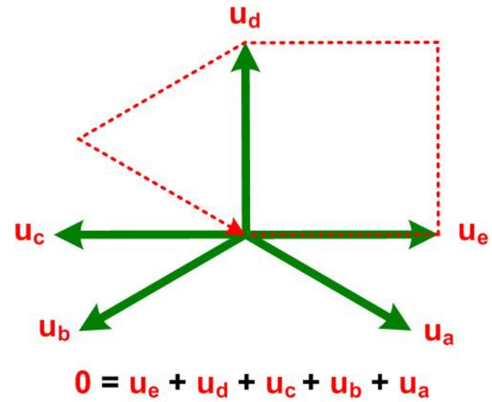


Figure 5 Equilateral pentagon unit vector sum = 0

Nevertheless, Figure 5 shows that the sum of the unit vectors is **0**, which means the perpendicular constant sum property holds. As we mentioned before, the unit vector head-to-tail sum provides an image of the original polygon rotated 90°. *So if the original polygon is equilateral, then the unit vector sum image will also be a closed convex polygon and so yield a zero sum result.*

References

[1] Stevenson, Jim, "Polygon Altitude Problems I," 27 August 2018