Perspective Map

(18 July 2016, rev 8 Aug 2016, rev 2 January 2019)

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A number of recent puzzles have involved perspective views of objects (see for example below p.8). I had never really explored the idea of a perspective map in detail. So some of the properties associated with it always seemed a bit vague to me. I decided I would derive the mathematical equations for the perspective or projective map and see how its properties fell out from the equations.

Figure 1 Shows the basic idea. A person is standing at the origin (0, 0, 0) of a three dimensional axis system with the z-axis being vertical and the y-axis straight out in front, perpendicular to the x-axis. Their eyes are at height h above the xy-plane. They are looking at objects through a view screen located a distance d from their face. So sight-lines will associate points (x, y, z) on the objects to points (X, Y, Z) on the view screen (where always $Y = d$).

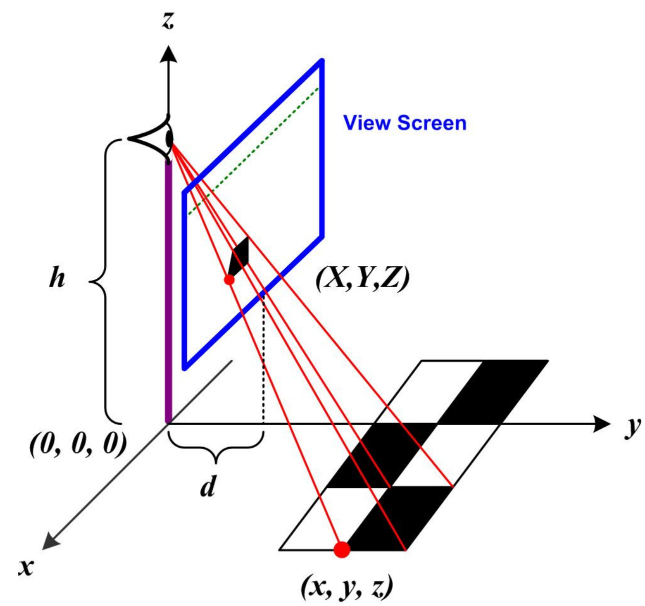


Figure 1 Perspective Lines

Figure 2 and Figure 3 provide views of the yz-plane and xy-plane, respectively, from which we can extract the values of X and Z that correspond to the projected point on the view screen.

From Figure 2 we get the relation

$$\frac{Z - z}{y - d} = \frac{h - z}{y}$$

which gives Z in terms of y and z, namely,

$$Z = z + \frac{(h - z)(y - d)}{y} = h - (h - z)\frac{d}{y}$$

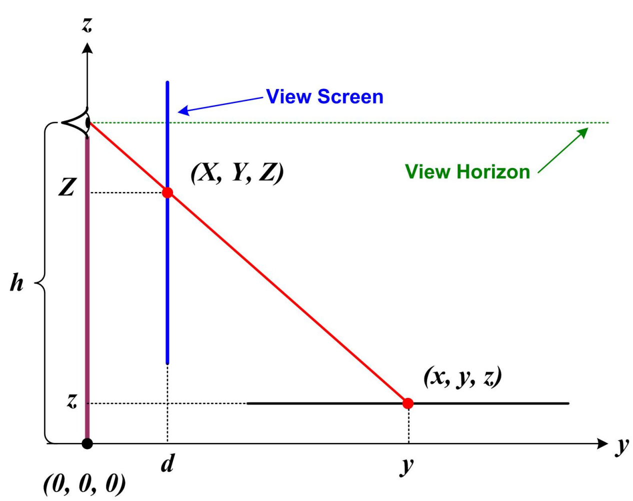


Figure 2 xz View

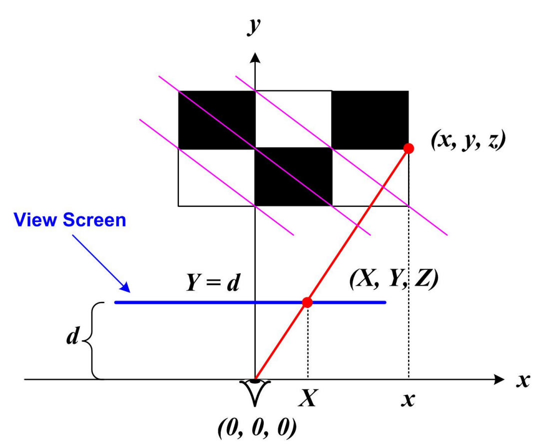


Figure 3 xy View

From Figure 3 we get

$$\frac{X}{d} = \frac{x}{y} \quad \Rightarrow \quad X = d\,\frac{x}{y}$$

So the perspective map $(x, y, z) \rightarrow (X, Y, Z)$ is given by

Perspective Map

$$\boxed{\begin{aligned} X &= x\frac{d}{y} \\ Y &= d \\ Z &= h-(h-z)\frac{d}{y} \end{aligned}} \tag{1}$$

Let us now consider what happens to sets of parallel lines under the perspective map. We will be looking at lines parallel to each of the axes and also lines perpendicular to the z-axis but rotated from the strictly horizontal or perpendicular. Figure 4 shows the results for all the parallel lines except those parallel to the z-axis. The details will be discussed below.

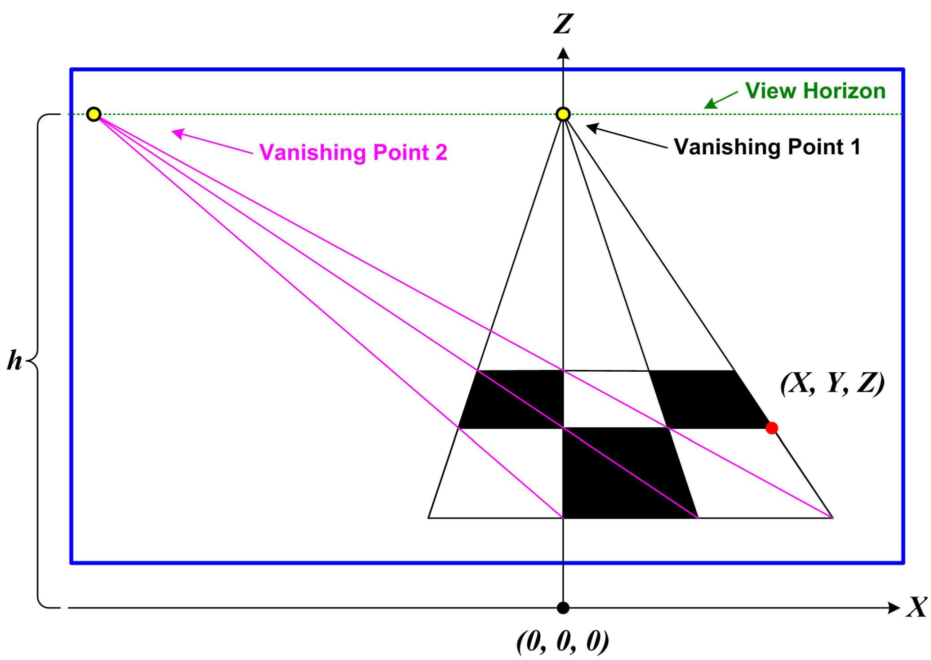
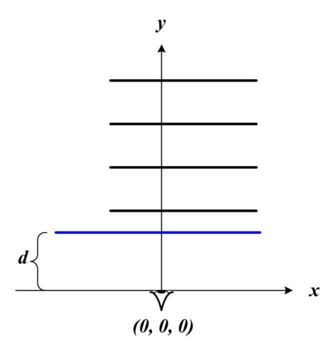


Figure 4 View Screen showing projections of various sets of parallel lines.

Horizontal Parallel Lines

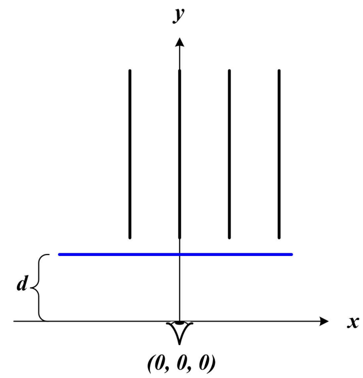
First we consider lines parallel to the x-axis. This means the y and z coordinates are constant and only the x coordinate varies. We see from equation (1) that X varies linearly with x and Z remains an unchanged constant. So the resulting projections on the view screen will be horizontal parallel lines at various fixed elevations Z.



Vertical Parallel Lines

Consider next lines parallel to the z-axis. In this case, x and y are held constant and z varies. From equation (1) X and Y are fixed and only Z varies linearly with z, which just amounts to a reparameterization of the line. So the parallel vertical lines remain parallel and vertical in the view screen.

Perpendicular Parallel Lines



Consider next parallel lines perpendicular to the view screen (parallel to the y-axis). This is where things get interesting. Here we fix x and z and let y vary. In order to view only images in front of the view screen, we will assume $y > d$. From equation (1) as $y \to \infty$, we see that $X \to 0$ and simultaneously, $Z \to h$. This point $(0, d, h)$ is labeled Vanishing Point 1 in Figure 4. So note that *all* the perpendicular parallel lines end up at this vanishing point on the horizon. Note further that this point is directly opposite the line of sight of the observer standing at (0, 0, 0).

In our figures we have been tacitly assuming $z < h$, so that the parallel lines approach the vanishing point from below. But there is no reason to make such a restriction, that is, we can have $z > h$ and still the lines will converge to the same vanishing point. Figure 5 from the article on Dürer's table to be discussed below (p.4) shows lines emanating from points z above the horizon h.

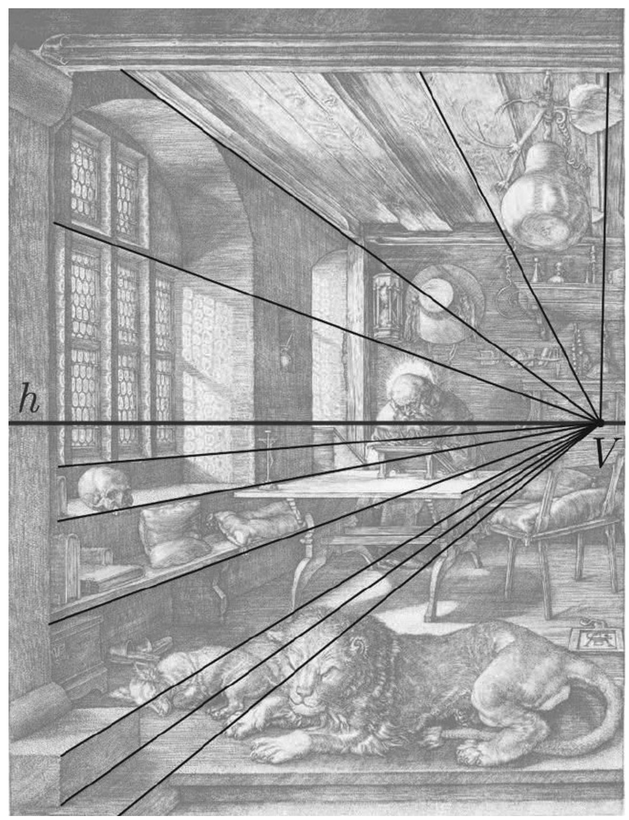


Figure 5 ***St. Jerome in His Study*, by Albrecht Dürer, 1514. The vanishing point V for lines that are perpendicular to the picture plane. The height of the vanishing point gives us the horizon line h. (Crannell et al.)**

Rotated Horizontal Parallel Lines

Consider next parallel lines rotated and perpendicular to the z-axis. Things become even more interesting in this case.

Suppose the lines are rotated by an angle θ with respect to the y-axis as shown in Figure 6. Such a rotation takes a point (x, y) (ignoring the constant z) and maps it to the point (x', y') via the equations

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \end{aligned} \qquad (2)$$

Now equations (1) become

$$X = d\frac{x'}{y'}, \quad Y = d, \quad Z = h - (h - z)\frac{d}{y'}$$

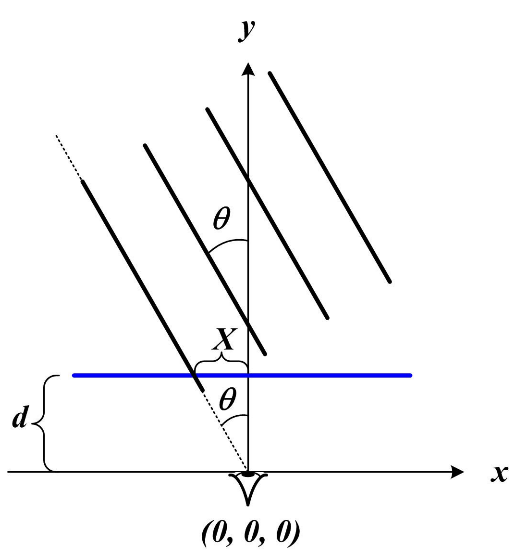


Figure 6 Rotated Parallel Lines

Since every set of rotated parallel lines is obtained from a set of perpendicular lines, we can fix $x = c$ a constant to select a particular line and then let y vary. This has the effect of parameterizing the (x', y') line by y. Then we have as $y \to \infty$,

$$X = d\,\frac{x'}{y'} = d\,\frac{c\cos\theta - y\sin\theta}{c\sin\theta + y\cos\theta} = d\,\frac{c\cos\theta / y - \sin\theta}{c\sin\theta / y + \cos\theta} \to -d\,\frac{\sin\theta}{\cos\theta} = -d\tan\theta$$

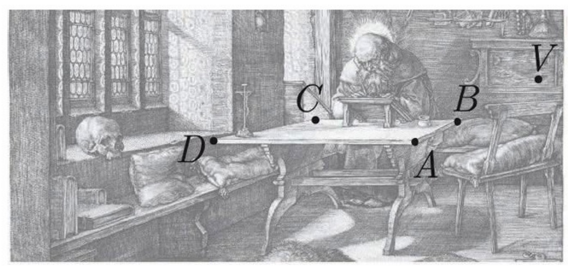
$$Z = h - (h - z)\frac{d}{y'} = h - (h - z)\frac{d}{c\sin\theta + y\cos\theta} \to h$$

So all the rotated parallel lines converge on a vanishing point (e.g. Vanishing Point 2 in Figure 4) on the horizon line $\boldsymbol{h}$ at a distance $\boldsymbol{d}$ **tan** $\boldsymbol{\theta}$ to the left of the perpendicular lines' vanishing point. The point is to the left for counter clockwise or positive rotations θ and to the right for clockwise or negative rotations. Notice that this second vanishing point depends on the distance the view screen is from the observer. This property will be essential for the understanding of the Dürer article below.

Dürer Table Controversy

The article "Dürer: Disguise, Distance, Disagreements, and Diagonals!" by Annalisa Crannell, Marc Frantz, and Fumiko Futamura,[1] addresses a controversy over Albrecht Dürer's woodcut *St. Jerome in His Study* (1514) shown in Figure 5. William Mills Ivins Jr., curator of the department of prints at New York's Metropolitan Museum of Art from 1916 to 1946, criticized Dürer's use of perspective in the image. He said, "The top of the saint's table is of the oddest trapezoidal shape—certainly it is not rectangular." Crannell and company took issue with Ivins' criticism and claimed Dürer was perfectly accurate in showing the perspective for a square table. They said, "The oddness that Ivins saw in the table wasn't because *Dürer* was in the wrong, but because Ivins was in the wrong, literally: he was looking from the wrong place!"

Figure 7 demonstrates that if a viewer stands in the wrong place for such a highly perspective image, the objects seem distorted. The rest of the article discusses (but does not explain) where the viewer should stand to see the image correctly.



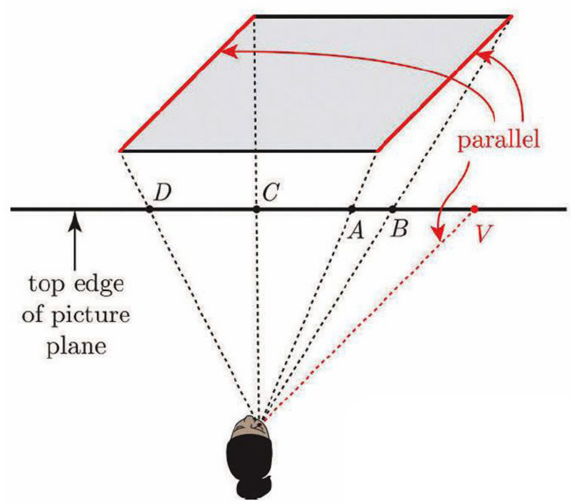


Figure 7 If we stand in front of C, the table does not appear to be square—otherwise, how could we see the side AB? (Crannell et al.)

As shown above in Figure 5, Crannell et al. find the vanishing point V for lines perpendicular to the plane of the image, in particular, for the left and right edges of the table. They had claimed the table is not rotated relative to the viewer, so that the edges would be perpendicular to the plane of the picture. They then draw a line from the lower right corner of the table top A to the upper left corner C and project it to the visual horizon h at the second vanishing point labeled Z (Figure 8). They designate the distance between the two vanishing

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1 8 November 2014: : Math Horizons: : www.maa.org/mathhorizons (http://www.maa.org/sites/default/files/pdf/horizons/durer.pdf)

points, V and Z, as d. Finally they say to see the picture properly, the viewer must stand with one eye directly in front of the first vanishing point V and a distance d from the picture (Figure 9). **Why?**

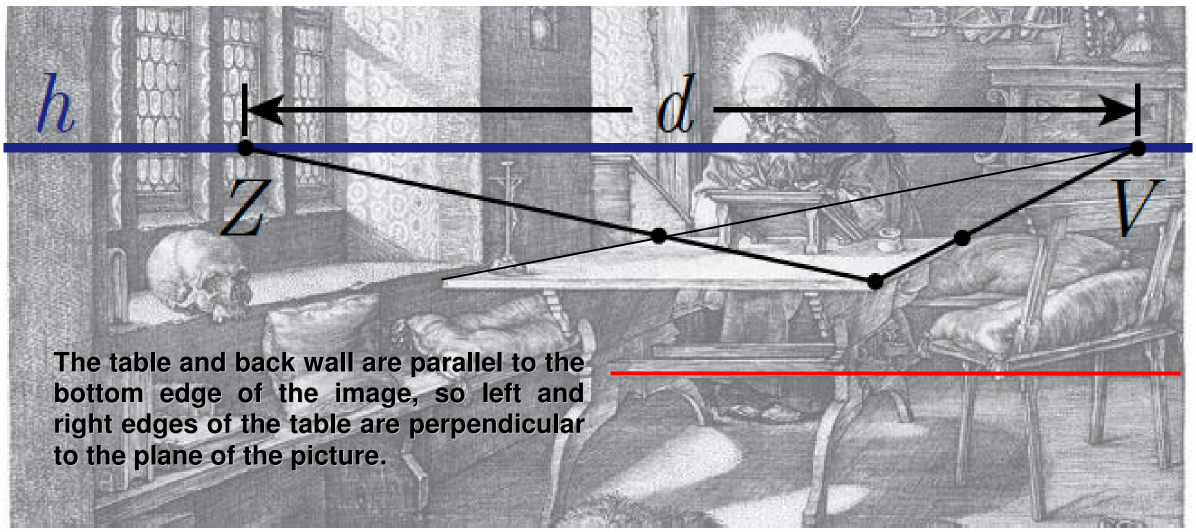

Figure 8 The diagonal across the top of the table has a vanishing point Z on the horizon. (Crannell et al.)

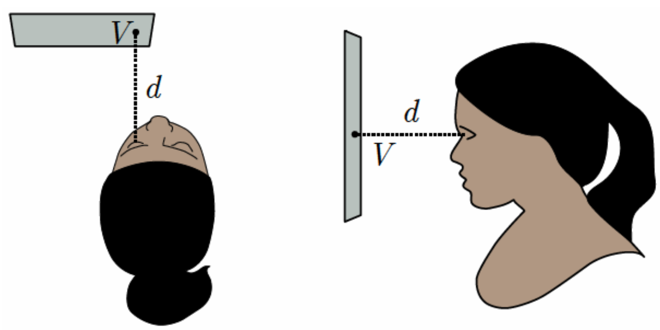

Figure 9 To look at *St. Jerome in His Study* so that it appears most three-dimensional, view it with one eye in front of the point *V*, at a distance *d* from the picture. (Crannell et al.)

The fact that the edges of the table are perpendicular to the plane of the picture (our view screen) means our discussion above applies. So the rotated line (table diagonal) has a vanishing point Z located a distance $d \tan \theta$ from the vanishing point V where d is the distance from viewer to picture (Figure 10). The assumption that the table is square means the angle of rotation for the diagonal is $\theta = 45°$ so that $\tan \theta = 1$ and $d \tan \theta = d$. So if a viewer of the picture stands a distance d from the picture squarely in front of the vanishing point V, then they are in the exact location of the "observer"

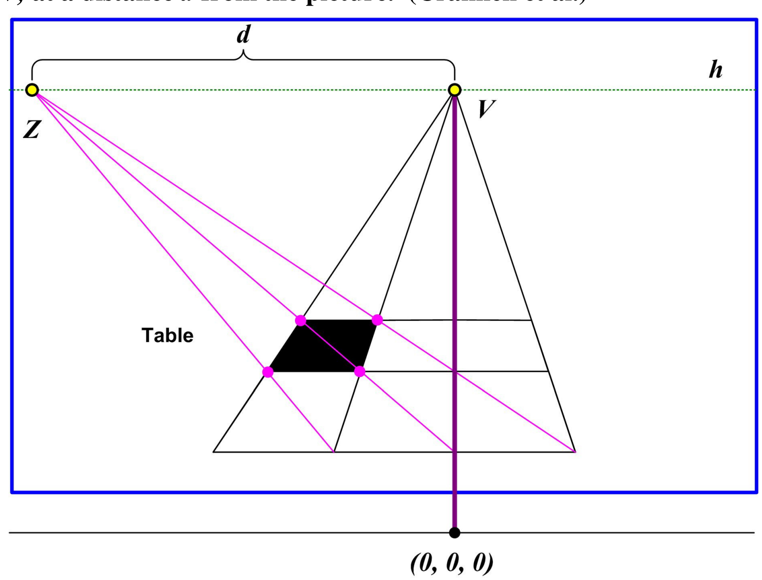

Figure 10 Dürer's Table

who generated the lines. That is, if the objects in the picture were real, then a person standing at the indicated location would see the objects just as they were viewed in the picture.

One minor point. If the table were not square and had the long side facing front, then the rotation of the diagonal line would have been greater than 45° and the tangent greater than 1. This would mean the distance the viewer would have to stand in front of the picture would have to be less than the separation between the vanishing points.

Evelyn Lamb has an amusing blog in her *Scientific American* Roots of Unity website[2] about touring the Brigham Young University Museum of Art with Annalisa Crannell. Crannell aligned chopsticks over pictures to try to find vanishing points and the proper position to stand to view the picture correctly. When successful, apparently the images almost appeared three-dimensional.

Perpendicular Assumption

The assumption that one of the sets of parallel lines is perpendicular to the view screen is essential for Crannell's procedure to work. In the photo I took of the Washington National Cathedral in 2011 shown in Figure 11, none of the lines in the photo were originally perpendicular to the plane of the photo, so the viewpoint should not have been in front of one of the two vanishing points. Of course the viewpoint was actually on the bisector of the photo, as shown by the yellow dot.

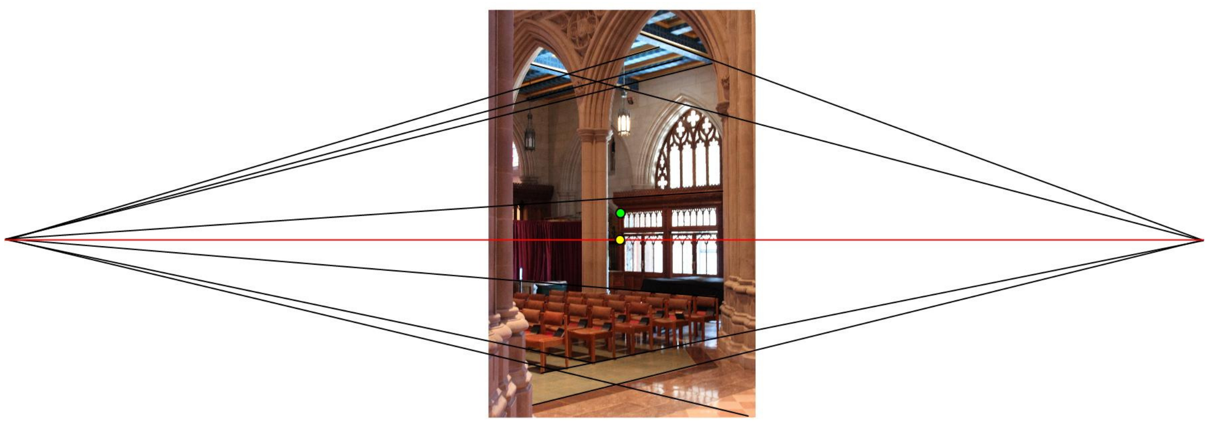

Figure 11 Rotated, Non-perpendicular Lines

Note that the red horizon line is slightly below the center of the photograph shown by the green dot. This means that the camera was angled a slight bit upwards. This is tantamount to saying the view screen is slightly tilted away from the vertical. We shall explore this situation in more depth in the following section.

Rotated Vertical Parallel Lines

Figure 12 shows a case where the view screen (camera) had been rotated upwards at a considerable angle. The horizon line indicated by the downward slanting ray lines to the pair of vanishing points is very far below the view screen. But the vertical lines following the pillars of the cathedral converge at a third vanishing point above the view screen.

We need to modify our equations to handle the case where xyz-space of Figure 1 has been rotated about the x-axis. Figure 13 shows how we can represent the xz view (Figure 2) with a virtual observer standing at (0, 0, 0) but canted back at an angle ϕ and shrunk to a height $h' = h \cos \phi$. This virtual observer will now be a distance $d' = d + h \sin \phi$ from the view screen.

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2 "How to Look at Art: a Mathematician's Perspective" 28 April 2016 (http://blogs.scientificamerican.com/roots-of-unity/mathematical-perspective-in-art/, retrieved 4/29/2016)

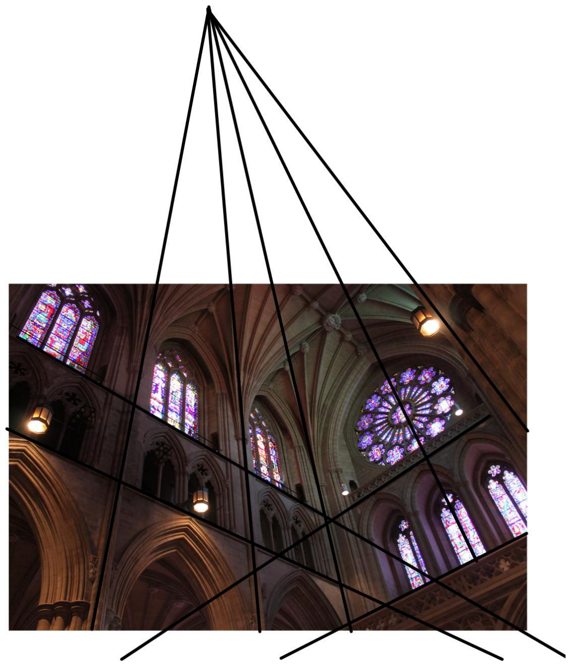


Figure 12 Vertical Rotated Parallel Lines

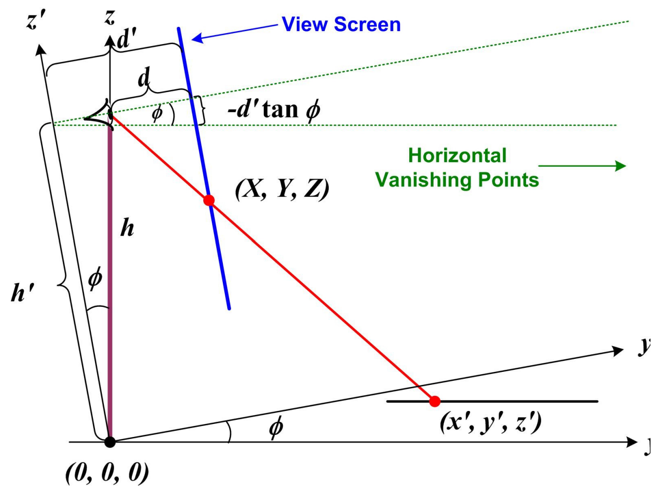


Figure 13 Rotated View Screen and xz View

We can now apply equations (2) to the rotation of the *y, z* coordinates (*x* fixed) to yield

$$y' = y \cos -\phi \;-\; z \sin -\phi$$
$$z' = y \sin -\phi \;+\; z \cos -\phi$$

where the rotation of the y'z' view by ϕ causes the original yz view to rotate $-\phi$. Thus we have

$$y' = y \cos \phi \;+\; z \sin \phi$$
$$z' = -\; y \sin \phi \;+\; z \cos \phi \qquad (3)$$

So the projection or perspective map becomes $X = d'\dfrac{x'}{y'}, \quad Y = d', \quad Z = h' - (h' - z')\dfrac{d'}{y'}$

We can see what happens to a horizontal straight line that is perpendicular to the x-axis when the view screen is rotated counter-clockwise by an angle ϕ. Make x and z constant to select a particular line and then let y vary. This has the effect of parameterizing the (x', y', z') line by y. Then we have as $y \to \infty$ (noting that $h' = h \cos \phi$. and $d' = d + h \sin \phi$ are constant),

$$X = d'\frac{x'}{y'} = d'\frac{x}{y\cos\phi + z\sin\phi} \to 0$$
$$Y = d' \qquad (4)$$
$$Z = h' - (h' - z')\frac{d'}{y'} = h' - h'\frac{d'}{y'} + d'\frac{z'}{y'} = h' - h'\frac{d'}{y\cos\phi + z\sin\phi} + d'\frac{-y\sin\phi + z\cos\phi}{y\cos\phi + z\sin\phi} \to h' - d'\tan\phi$$

And so the line of vanishing points for the horizontal lines perpendicular to the x-axis (horizon) drops vertically below the observer line of sight perpendicular to the view screen by a distance (on the view screen) of $\boldsymbol{d'\tan\phi}$ and moves horizontally to the center.

Now consider what happens to the equations (4) when a rotated vertical line is considered. Select a vertical line by holding x and y constant and letting z vary. This parameterizes the line by z. As

$z \to \infty$, $X \to 0$ again, $Y = d'$ again, but this time $Z \to h' + d' \cot \phi$. But

$$\cot \phi = \tan (90° - \phi)$$

So the vertical vanishing point occurs at $X = 0$, $Z = h' + d' \tan (90° - \phi)$. This is illustrated in Figure 14.

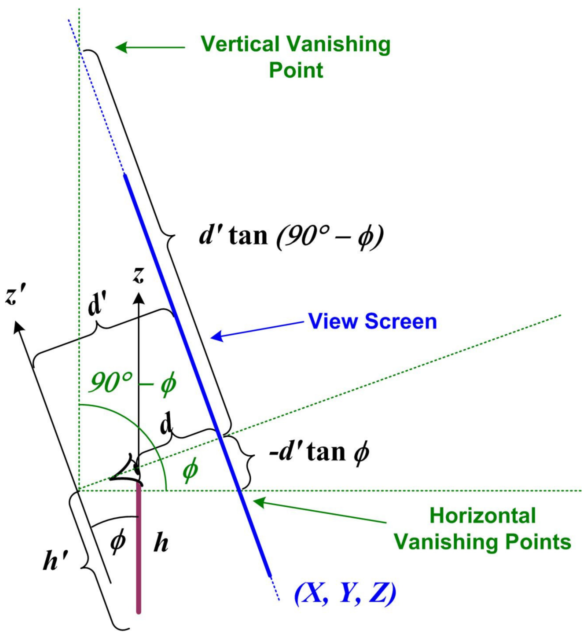


Figure 14 Rotated View Screen with Vanishing Point Locations

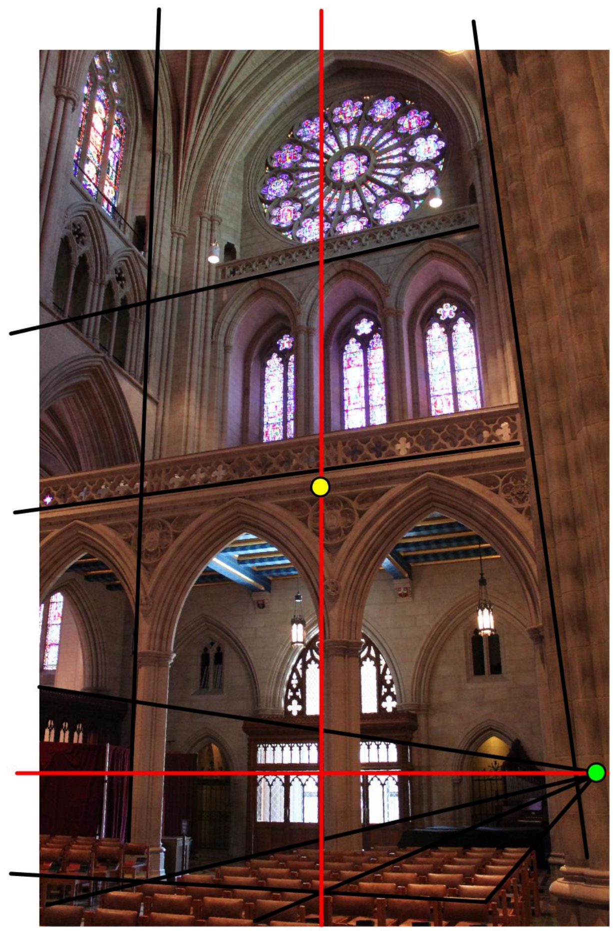


Figure 15 Vertical Rotated Parallel Lines

A concrete example of the effects of tilting the view screen (camera) is shown in Figure 15. The center of the observer view point (center of the photo) is shown as a yellow dot. The horizontal red line is the horizon on which lie the vanishing points of the horizontal (rotated) parallel lines. One of these vanishing points is shown as a green dot. The vertical red line through the center of the photo holds the vanishing point for the vertical parallel lines (cathedral columns). Given the inclination ϕ of the camera, the horizon line is considerably below the center view point.

Futility Closet Puzzle

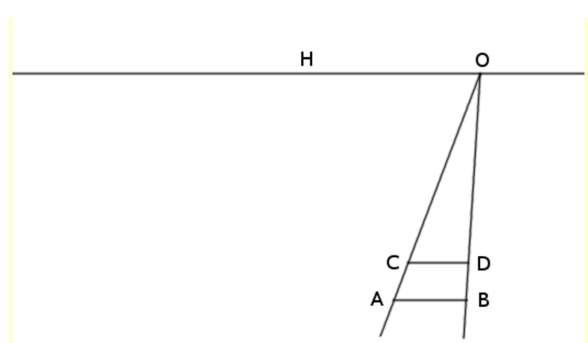
As an application of these ideas about perspective, consider the following puzzle.

(http://www.futilitycloset.com/2016/04/01/perspective-11/, retrieved 8/8/16)

Perspective

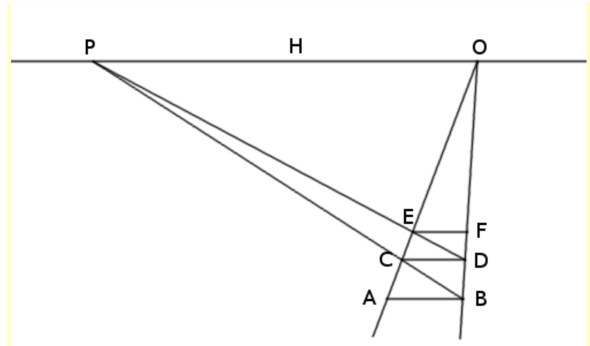
(1 April 2016)

AB and CD are consecutive ties across a pair of railroad tracks that appear to meet at O on the horizon, H. If the ties are parallel to the horizon and are equally spaced along the tracks, how can we draw the next tie in this perspective figure?



Solution

In the figure, parallel lines meet at the horizon, which represents infinity. If all the ties are evenly spaced, then the railroad track is a series of identical rectangles, and the diagonals of these rectangles are parallel. If diagonal BC reaches the horizon at P, then the line through D that's parallel to it is PD. That gives us E, the point where the next tie meets the left track, and all that remains is to draw EF parallel to the horizon.



(From Ross Honsberger, *More Mathematical Morsels*, 1991.)

Based on our discussion above (see Figure 10), this solution makes sense and is quite ingenious.

Addendum

2 January 2019

I read somewhere that a parabola under a perspective map becomes an ellipse. Certainly the lines of the parabola, as they head off to infinity, should converge to a point on the view horizon, and thus form a closed curve. But is it exactly an ellipse? So with laborious calculations (and my numerous arithmetic mistakes) I finally was able to show it was true. Not to have my efforts go to waste, I thought I would include the results as an addendum to this article.

Lay out a simple parabola on the xy-plane (z = 0). Have the view screen rest on the xy-plane at the distance y = d from the observer. Then the parabola

$$y = x^2 + d$$

will have its vertex at (0,d,0). The perspective map equations become

$$X = d\frac{x}{x^2+d}, \quad Y = d, \quad Z = h\left(1-\frac{d}{x^2+d}\right) \tag{5}$$

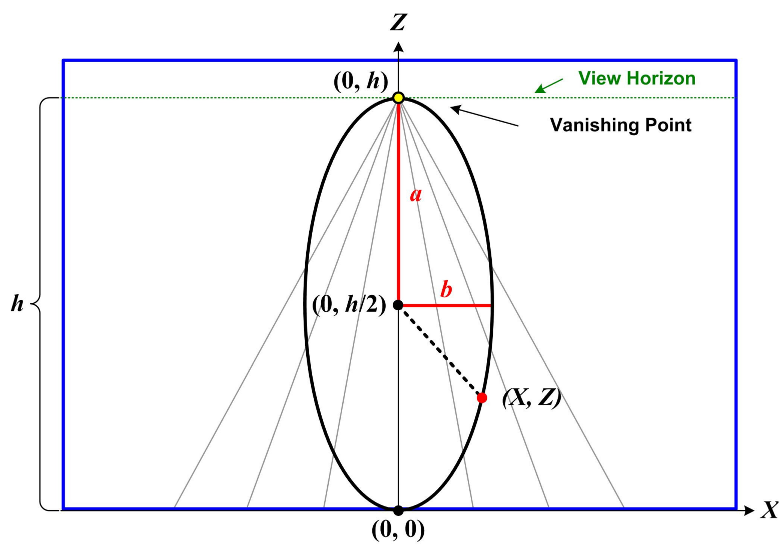


Figure 16 Ellipse as Projected Parabola

To show the parabola becomes an ellipse, as in Figure 16, we want to show

$$\frac{(X-0)^2}{b^2} + \frac{\left(Z-\frac{h}{2}\right)^2}{\left(\frac{h}{2}\right)^2} = 1 \tag{6}$$

where the semimajor axis $a = h/2$ and the semiminor axis b is yet to be determined.

Semiminor axis *b*. If the figure is to be an ellipse, then the distance b represents the maximum distance along the X-axis from the Z-axis. So we will compute the derivative dX/dZ and set it to 0.

$$\frac{dX}{dZ} = \frac{\frac{dX}{dx}}{\frac{dZ}{dx}} = \frac{x^2 - d}{2xh}$$

The derivative vanishes when $x = \sqrt{d}$. From equation (5) this means $X = \sqrt{d}/2$ and $Z = h/2$. So the maximum excursion b also occurs at the center of the putative ellipse, which bodes well for the eventual demonstration.

Thus, taking $b = \sqrt{d}/2$ to be our semiminor axis, equation (6) becomes

$$\frac{(X - 0)^2}{\frac{d}{4}} + \frac{\left(Z - \frac{h}{2}\right)^2}{\left(\frac{h}{2}\right)^2} = 1$$

From equation (5) the left hand side (LHS) becomes

$$\frac{4}{d}\left(\frac{xd}{x^2 + d}\right)^2 + \frac{4}{h^2}\left(h\left(1 - \frac{d}{x^2 + d}\right) - \frac{h}{2}\right)^2$$

which after some tedious arithmetic (and many screw-ups) reduces to 1, which is what we wanted to show. So the perspective view of a parabola is an ellipse.

(This exercise also gives me confidence that I figured out the correct perspective map equations.)