Perimeter-Area Problem

6 January 2019

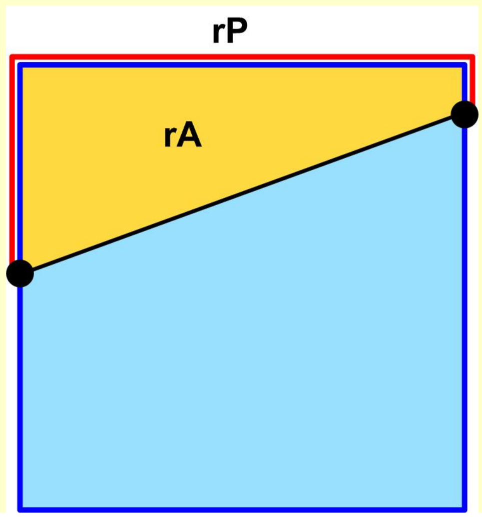
Jim Stevenson

James Tanton posted the following interesting problem on his Twitter account:

James Tanton, 6 January 2019 06:26 AM

(https://twitter.com/jamestanton/status/1081919893459546112)

A square of area A, perimeter P. For which values 0< r <1 is there a line across the square that chops off rA of the area and rP of the perimeter as shown? (r = 1/2, yes; r = 1/3, no.)



Solution

Unfortunately the only approach I could find involved an argument by cases, which I generally dislike. There may be a more elegant solution, but it was not immediately obvious.

There are a number of symmetries that reduce the scope of the problem. First, let P' = rP and A' = rA. Then r = P'/P = A'/A. Notice that the complementary perimeter and area also satisfies equal ratios if the originals did. That is, (P – P')/P = 1 – r = (A – A')/A. And finally, *any* line through the center of the square divides the square and its perimeter into congruent pieces and therefore of equal length and area, respectively. In which case, such a line satisfies the case P'/P = A'/A = 1/2 (Figure 1). Thus, we only need to consider cases for which the area A' does not contain the center of the square.

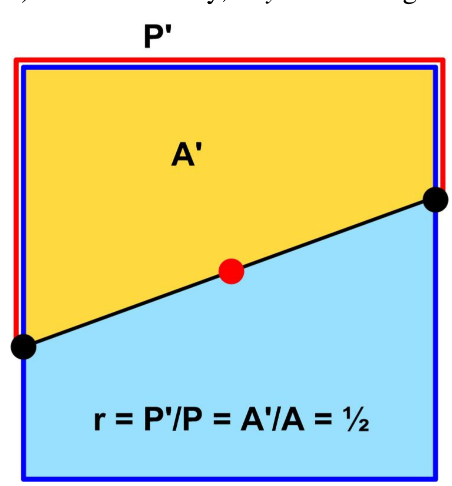


Figure 1 Line Through Center ⇒ r=1/2

We shall now consider two cases as shown in Figure 2 and Figure 3 where we have assigned the length 1 to the side of the square without any loss of generality. By the mirror symmetry about the vertical line through the center of the square, these two cases suffice. We let s_0 be the distance of a fixed point along the left edge of the square from the top edge. We let s be the distance of a variable point along the top edge from the left edge.

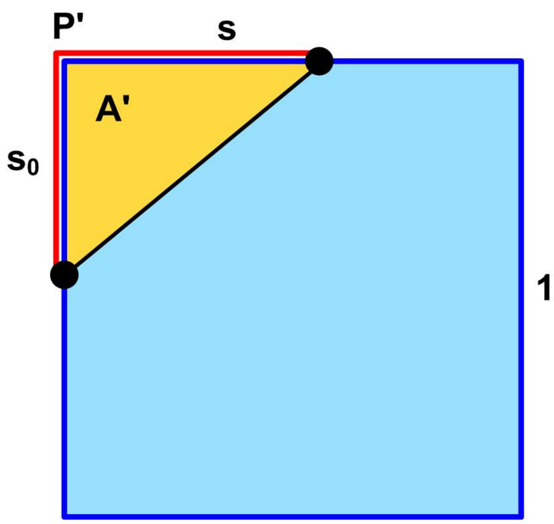


Figure 2 **Case1: $0 < s \le 1$, $0 < s_0 \le 1$, $s + s_0 < 2$**

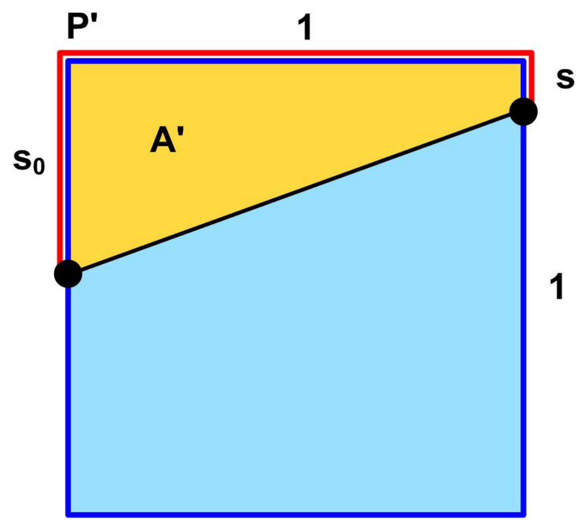


Figure 3 **Case 2: $0 < s \le s_0 \le 1$, $s + s_0 < 1$**

Case 1: $0 < s \le 1$, $0 < s_0 \le 1$, $s + s_0 < 2$ (Figure 2)

We constrain $s + s_0 < 2$, since $s + s_0 = 2$ represents a diagonal line through the center, and we already know that satisfies $r = ½$. Then $P = 4$ and $P'/P = (s_0 + s)/4$, and $A = 1$ and $A'/A = (1/2)s_0s$. Equating the two ratios yields

$$(s + s_0) = 2s_0s$$

or

$$s = s_0/(2s_0 - 1)$$

But $s \le 1$ means $s_0 \le (2s_0 - 1)$ or $1 \le s_0$. But $s_0 \le 1$, so $s_0 = 1$. That means $s = 1$, or $s + s_0 = 2$, which has been disallowed. So no new value for r has been found in this case.

Case 2: $0 < s \le s_0 \le 1$, $s + s_0 < 1$ (Figure 3)

We constrain $s + s_0 < 1$, since $s + s_0 = 1$ again represents a diagonal line through the center, and we have already considered that. Again by the mirror symmetry about a vertical line through the center of the square, we can assume $s \le s_0$. Then $P'/P = (s_0 + 1 + s)/4$, and $A'/A = s + (1/2)s(s_0 - s)$. Equating the two ratios eventually yields the quadratic

$$2s^2 - (3 + 2s_0)s + (s_0 + 1) = 0$$

which in turn, after a bunch of algebra, produces two potential solutions:

$$s = s_0 + 1 \text{ or } s = ½.$$

The first solution means $s > 1$, but we require $s \le 1$, so this solution is eliminated. If $s = ½$, then $s \le s_0 \le 1$ means $½ \le s_0 \le 1$. But $s + s_0 < 1$ implies $s_0 < ½$. Hence, this solution is also eliminated. Therefore again no new value for r has been found for this case.

Between Case 1 and Case 2 and all the symmetric versions, we have eliminated the equality of the ratios for any r other than ½ and for any line through the square other than one through the center. So the final answer is $r = ½$ yes and no for any other value of $0 < r < 1$.