Meditation on "Is" in Mathematics Part II – Mathematical Reality

(7 February 2015, rev 26 April 2017, rev 17 January 2018)

Jim Stevenson

The following is a continuation of my meditation on the nature of mathematics as I see it. In Part I, I wrote a detailed critique of a March 2014 article I read in *Slate* attempting to popularize mathematical concepts associated with Zeno's Paradox.[1] In this Part II, I will try to explain in more depth the admittedly philosophical concepts behind my remarks about that article.

The Reality of Mathematics

The heavily annotated discussion about Zeno's paradox and its resolution raised again the issue of the "reality" of mathematics, that is, is it something we discover or do we invent it?[2] This arose particularly over the statement in the article that "1/2 + 1/4 + 1/8 + 1/16 ... adds up to 1," in other words, the infinite sum *is* 1. What do we mean by *is*?

Mathematics as Invention

The thrust of my annotations appears to weigh in on the side of imaginative invention, especially my quote from Bressoud:[3] "… mathematicians are not using definitions as they are usually encountered, as descriptions of entities that already exist. For mathematicians, definitions are prescriptive." This is also captured well by a quote from G. H. Hardy (*Divergent Series*, 1949) found in Ellenberg's book:[4]

> It does not occur to a modern mathematician that a collection of mathematical symbols should have a "meaning" until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say, in so many words, "by X we *mean* Y." … It is broadly true to say that mathematicians before Cauchy asked not, "How shall we *define* 1 - 1 + 1 - 1 + ..." but "What *is* 1 - 1 + 1 - 1 + ...?" and that this habit of mind led them to unnecessary perplexities and controversies which were often really verbal.

It would be safe to say regarding this issue that most non-mathematicians (students, general populace) are at the 18th century stage, at best, whereas virtually all 20th and 21st century mathematicians are at the modern stage. This becomes a barrier for professional mathematicians to teach mathematics to novices. When I was a student starting out in real mathematics in high school, especially when I glanced at an appendix in my Algebra II book that discussed infinite series, I

---

1 Brian Palmer, "What Is the Answer to Zeno's Paradox?" *Slate*, 5 March 2014, (http://www.slate.com/articles/health_and_science/science/2014/03/zeno_s_paradox_how_to_explain_the_solution_to_achilles_and_the_tortoise.html, retrieved 2/5/15)

2 This is actually a long-running question about mathematics. See, for example, Leslie A.White, "The Locus of Mathematical Reality: An Anthropological Footnote," *Philosophy of Science*, October, 1947, in James R. Newman, *The World of Mathematics*, Volume 4, George Allen & Unwin, 1956 (https://archive.org/download/TheWorldOfMathematicsVolume1/Newman-TheWorldOfMathematicsVolume4_text.pdf) pp.2348-2364.

3 Quoting Barbara Edwards, *Undergraduate Mathematics Majors' Understanding and Use of Formal Definitions in Real Analysis*. The Pennsylvania State University, State College, PA, 1997. PhD thesis

4 Jordan Ellenberg, *How Not to be Wrong: The Power of Mathematical Thinking*, Penguin Press, New York, 2014, p.47.

reacted with disbelief — adding up an infinite number of positive numbers to yield a finite result was total nonsense. To be told it was "true" made me feel stupid for not seeing it. In fact, at the 5th century BC stage of Greek mathematics it is nonsense — it literally does not make sense. It took mathematicians some 2000 years to make sense of it, that is, make up a meaningful way to assign a finite value to an infinite sum. I only came to terms with the astounding statements in mathematics when I realized they ultimately came from the minds and imaginations of people and were not particularly evident without instruction.

If mathematical concepts were made up, why believe them? OK, what do we mean by believe? Well, "accept" might be a better term than "believe." Hovering around this discussion is also the word "true," that is, how can we accept mathematical statements as being true.

Basically a system of rules was established by the Greek mathematicians that were consolidated by Euclid (fl. 300 BC) into his books called the *Elements*. It provided a system of deductive proof (mostly involving statements about plane geometry) that would establish the truth of a new statement based on the (proved = derived) truth of previous statements and on a set of axioms. The axioms were a small set of "unproved" assumptions, that is, these were statements we just accepted (as obvious?) and were not the consequence of a deductive proof. So everything ultimately derives from axioms. Mathematics is just one massive collection of deductive chains of reasoning from a (small) basic set of statements.

We would like to say a mathematical statement is true if it can be proved, that is, derived via one of these chains of reasoning. There are some other ways of defining true statements. One way comes from the concept of "truth tables" that I won't go into. Another (weaker) way is via a "conjecture." This is a statement that mathematicians believe to be true, but they haven't figured out a proof yet. An example is the famous Goldbach's conjecture, which states that every even integer greater than 2 can be expressed as the sum of two primes (e.g. $8 = 3 + 5$, $12 = 5 + 7$, $20 = 3 + 17 = 7 + 13$, …).

So we at least want all statements derived (proved) from axioms to be true. A set of axioms is said to be *consistent* if only true statements can be derived from them. If a false statement (contradiction) could be derived (proved), then it is possible to show any statement can be derived from the axioms, and so the whole effort is worthless, that is, we don't know what is true or false from a proof.

On the other hand, it would also be great if we could know that *all* true statements (determined by whatever means) could be derived from our set of axioms. Such a set of axioms with this property is said to be *complete*: all true statements (however arrived at) can be proved from these axioms. We just saw that an inconsistent set of axioms is complete, since any statement, true or false, can be derived from them.

The great **Incompleteness Theorem** of Kurt Gödel (1931)[5] essentially says any set of axioms rich enough to produce the rules of arithmetic cannot be both consistent and complete, that is, there will always be a statement P that cannot be derived from the axioms. Astoundingly, that statement P is effectively "This set of axioms is consistent." (The self-referential nature of the statement is not an accident.) So not only can we not *prove* (using the given set of axioms) that all true statements that exist can be derived from the axioms, we cannot even *prove* (again using the given set of axioms) that this set of axioms will only yield true statements.

The bottom line here is that the ultimate goal of mathematics — to prove its own consistency, that only true statements can be derived — is doomed to failure. So the idea that mathematically derived

[5] For the best introduction see *Gödel's Proof* by Ernest Nagel and James Newman (1958). A more modern treatment can be found in Torkel Franzen, *Gödel's Theorem: An Incomplete Guide to Its Use and Abuse,* 2005, with a reference to the use of Cantor's diagonalization argument on p. 70.

statements are to be believed or accepted as true is not as ironclad as we would like. But so far (over 3000 years) we have not encountered any inconsistencies (contradictions), unless you count paradoxes. But then, so far, all paradoxes have been resolved by further extensions to yield consistent mathematical constructs.

Mathematics as Discovery

Mathematics is different from the physical sciences in that it does not evolve through physical experimentation and observation of subsequent behavior. It seems to proceed more from the imagination and creativity of human minds, as suggested in the previous section. I certainly am in sympathy with this idea, as argued in my comments to the Zeno Paradox article. On the other hand, there is something more constrained about mathematics compared with say an abstract painting or a piece of music (even though those media have their own constraints).

Platonism

Mathematical creativity is not unfettered, but, as we saw, must proceed along rigid lines of logical deduction. (There is a lot of discussion about mathematical ideas that spring suddenly into the mathematician's mind, which must then be anchored to the deductive chains of existing mathematical knowledge.) Given the constraints on mathematical development, it is perhaps not surprising that often mathematicians arrive at the same mathematical construct basically independent of one another. Usually these parallel "discoveries" are close in time (e.g. the calculus of Newton and Leibniz), suggesting that the body of mathematics has progressed to a point where the next level becomes more obvious. Other "discoveries" have been more spread out in time and among different cultures, such as the so-called Pythagorean theorem (the sum of the squares of the sides of a right triangle is equal to the square of the hypotenuse or $a^2 + b^2 = c^2$), which showed up in Babylonia long before it was associated with Pythagoras (of course this also gets into the issue of "diffusion" vs. "independent discovery").

This suggests that there is something permanent and "pre-existing" about mathematics, which is independent of the individual human mind. Given enough time, intelligence, and mathematical progress, the next phases of mathematical concepts are predetermined. From this view mathematics is not just the random creation of human imagination; it is something that the human mind is disposed to uncover, given enough support. It is there to discover, not to invent.

This notion of a pre-exiting, ultimate form of something independent of human rational activities that is manifested to us tangibly in a number of possible ways is the essence of Plato's (c.428 – c.348 BC) concept of Forms developed in his book *The Republic*. As such, it is referred to as *Platonism*. A number of mathematicians have professed to hold this platonic view of mathematics, for example, Edward Frenkel,[6] Roger Penrose,[7] and David Mumford.[8] As Mumford formulates it, it is "The belief that there is a body of *mathematical objects, relations and facts about them* that is independent of and unaffected by human endeavors to discover them."

Mathematics and Self-Organization

I have to admit that I am shifting from my unfettered imagination view of math and moving more towards the platonic view. But I have a slightly different take on it. I view the logical "if …, then …" connections that make up a proof and form the chains of inference that constitute the body of mathematics akin to the chemical linkages and interactions that lead to the molecular structures of living organisms. That is, if we raise the question of where the platonic Form of the body of

[6] Edward Frenkel, *Love & Math: The Heart of Hidden Reality*, Basic Books, 2013, p. 234

[7] Referenced in Frenkel's book as Roger Penrose, *The Road to Reality*, Vintage Books, 2004, p.15.

[8] David Mumford, "Why I am a Platonist," *Newsletter of European Math. Society*, December 2008, pp. 27-30 (http://www.dam.brown.edu/people/mumford/beyond/papers/2008e--Platonism-EMS.pdf)

mathematics came from, we are not far from the questions about where living organisms come from. A creationist view would say the body of mathematics Form came fully developed from a creator. An evolutionist view would claim the Form did not pre-exist, but was built up in an incremental fashion. Pure evolution would talk about randomness and natural selection, which is basically a trial-and-error process trying to optimize fitness criteria. This does not quite suit our situation.

Recent developments in genetic evolution, however, are actually tending away from a totally random view of the subject. The ideas of complexity theory, involving many interacting autonomous agents that lead to organized outcomes (self-organization), have been applied to organic chemistry at the molecular level. A marvelous book by Peter M. Hoffmann, *Life's Ratchet*,[9] shows in detail how organic molecules, following the laws of chemical interactions, can self-organize into "living" entities. So the final high-level organism is pre-ordained from the lower-level rules, and is not a random outcome. In a sense, then, the final organism already "exists," is inherent, in the low-level molecules. This is analogous to the final picture in a jig saw puzzle that "exists" inherently in the disassembled pieces.

I view the evolution of the body of mathematics in the same way. It will be incrementally revealed to us, not in a random or arbitrary way, but in a pre-determined form dictated by the interlocking chains of deductive inference. Of course, this also means that without the human mind to perform these deductions, there would be no mathematics, contrary to the platonic view.

Mathematics and Physical Reality — a Mystery

To me, there is something about this mathematical reality that is different from physical reality. I believe the sum of an infinite series is a mentally arrived-at construct, however pre-determined, and not just a physical "is". But that means we must confront a mystery. If mathematics does not inhere in physical reality but is a human mental overlay, again however pre-determined, then why does it capture physical behavior so well?

One answer given recently by Max Tegmark is "that our reality isn't just *described* by mathematics – it *is* mathematics."[10] But we will consider less extreme notions.

Eugene Wigner

Much has been written about this issue of mathematics and physical reality, especially by Eugene Wigner in his famous 1960 article[11] where he says "The first point is that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it."

Wigner first addresses the question of what is mathematics:

> The principal emphasis is on the invention of concepts. Mathematics would soon run out of interesting theorems if these had to be formulated in terms of the concepts which already appear in the axioms. Furthermore, whereas it is unquestionably true that the concepts of elementary mathematics and particularly elementary geometry were formulated to describe entities which are directly suggested by the actual world, the same does not seem to be true of the more advanced concepts, in particular the concepts which play such an important role in physics. … Most more advanced mathematical concepts, such as complex numbers, algebras,

---

[9] Peter M. Hoffmann, *Life's Ratchet: How Molecular Machines Extract Order from Chaos*, Basic Books, 2012

[10] Max Tegmark, *Our Mathematical Universe: My Quest for the Ultimate Nature of Reality*, Vintage Books, 2014, p.254.

[11] Eugene Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," in *Communications in Pure and Applied Mathematics*, vol. 13, No. I (February 1960)

> linear operators, Borel sets — and this list could be continued almost indefinitely — were so devised that they are apt subjects on which the mathematician can demonstrate his ingenuity and sense of formal beauty. … The great mathematician fully, almost ruthlessly, exploits the domain of permissible reasoning and skirts the impermissible. That his recklessness does not lead him into a morass of contradictions is a miracle in itself.

Wigner seems to be leaning toward the imaginative invention school here and not the Platonists.

Addressing the role of mathematics in physical theories, Wigner has this to say:

> It is true, of course, that physics chooses certain mathematical concepts for the formulation of the laws of nature, and surely only a fraction of all mathematical concepts is used in physics. It is true also that the concepts which were chosen were not selected arbitrarily from a listing of mathematical terms but were developed, in many if not most cases, independently by the physicist and recognized then as having been conceived before by the mathematician. It is not true, however, as is so often stated, that this had to happen because mathematics uses the simplest possible concepts and these were bound to occur in any formalism. As we saw before, the concepts of mathematics are not chosen for their conceptual simplicity … but for their amenability to clever manipulations and to striking, brilliant arguments. …
>
> It is difficult to avoid the impression that a miracle confronts us here, quite comparable in its striking nature to the miracle that the human mind can string a thousand arguments together without getting itself into contradictions, or to the two miracles of the existence of laws of nature and of the human mind's capacity to divine them.

Further Examples: Stewart Shapiro[12]

Ohio State University philosopher Stewart Shapiro relates a puzzling experience that a friend once encountered in a physics lab.

> "The class was looking at an oscilloscope and a funny shape kept forming at the end of the screen. Although it had nothing to do with the lesson that day, my friend asked for an explanation. The lab instructor wrote something on the board (probably a differential equation) and said that the funny shape occurs because a function solving the equation has a zero at a particular value. My friend told me that he became even more puzzled that the occurrence of a zero in a function should count as an explanation of a physical event, but he did not feel up to pursuing the issue further at the time.
>
> "This example indicates that much of the theoretical and practical work in a science consists of constructing or discovering mathematical models of physical phenomena. Many scientific and engineering problems are tasks of finding a differential equation, a formula, or a function associated with a class of phenomena. A scientific 'explanation' of a physical event often amounts to no more than a mathematical description of it, but what on earth can that mean? What is a mathematical description of a physical event?"

What right do we have to presume that the natural world will hew to mathematical laws? And why does the universe oblige us so graciously by doing so? Repeatedly, mathematicians have developed abstract structures and concepts that have later found unexpected applications in science. How can this happen?

[12] Taken from the article "Augury" in *Futility Closet*, 4 September 2014, http://www.futilitycloset.com/2014/09/04/augury/, retrieved 2/6/16

"It is positively spooky how the physicist finds the mathematician has been there before him or her."

— Steven Weinberg

"I find it quite amazing that it is possible to predict what will happen by mathematics, which is simply following rules which really have nothing to do with the original thing."

— Richard Feynman

"One cannot escape the feeling that these mathematical formulae have an independent existence and intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them."

— Heinrich Hertz

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve."

— Eugene Wigner

(From Stewart Shapiro, *Thinking About Mathematics*, 2000; also his paper "Mathematics and Reality" in *Philosophy of Science* 50:4 [December 1983].)

More Than Descriptive

There is another aspect of the wonder of the mathematical description of reality (often referred to as knowledge representation by the old artificial intelligence community of the 1980s) that is rather profound (and suggested by the Feynman quote above). A while ago I did a detailed explanation of how Kepler's Laws were equivalent to Newton's Law of Gravity, given Newton's Laws of Motion.[13] All the laws discussed had mathematical representations, even though they were referring to physical situations. And the derivations of the relationships between these laws, including the equivalences, were all done solely with mathematics and without any physical reasoning. The physical phenomena that Kepler observed and codified mathematically in his laws turn out somehow to be inherent in the mathematical properties of Newton's laws (of motion and of gravity), which also represent physical phenomena (see Figure 1).

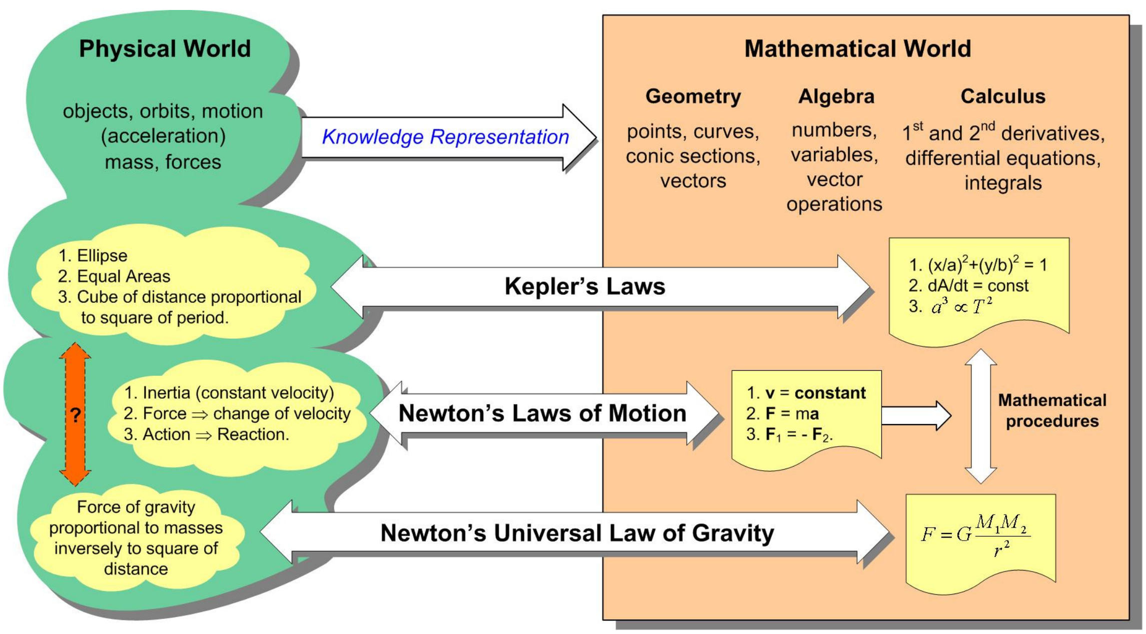

Figure 1 Mathematical Representation of Physical Reality

[13] This section is taken virtually verbatim from that previous article, "Kepler's Laws and Newton's Laws.".

I remember when I worked at the Geophysical Fluid Dynamics Laboratory there was an oceanographer who wrote down a particular form of the Navier-Stokes equations that represented the fluid motion of certain ocean regions. He manipulated the equations and produced successive transformations into alternative forms. When I asked him to show me the mathematical derivation, he admitted he did it "physically," that is, each term in the original equation had a physical meaning to him and he knew physically how these terms transformed into other terms that represented physical phenomena. In other words, to him the partial derivatives and other mathematical symbols in the equations were just labels or names of physical entities and he moved from one physical entity to the next using physical reasoning.

The derivation I did of the equivalence of Kepler's Laws and Newton's Law of Gravitation was not like that. It was all done mathematically. There was no physical mechanism indicated that would show how these two Laws were equivalent, only mathematical manipulations. So how is it that the mathematical if-then deductions relating purely mathematical entities mirror some invisible physical cause-and-effect logic that drives the physical results? "Mirror" is perhaps the wrong word, since *a priori* the mathematical deductive path is totally independent of the physical causal path. This is truly amazing.

Intelligent Equations

There is one other mysterious tie between the mathematical world and the physical world, suggested by the Heinrich Hertz quote above. The discovery of a mathematical representation of physical reality through some equations not only ties together currently observed phenomena into a logical whole, via the mathematics, but buried in the equations themselves are *mathematical* consequences that predict corresponding *physical* phenomena.

For example, when Maxwell produced his equations uniting electricity and magnetism into one electromagnetic phenomenon, a mathematical consequence was that the corresponding fields satisfied a wave equation. This implied there must be waves associated with electromagnetic disturbances, and sure enough Heinrich Hertz performed experiments to prove their existence. There are many more such examples as indicated by Tom Siegfried:[14]

> General relativity [equations] provided more surprises than just gravity waves, for instance. Black holes, gravitational lensing and even, in a way, the expansion of the universe emerged from Einstein's equations before any astronomer observed them. Quarks, the constituents of protons and neutrons, showed up in Murray Gell-Mann's math before evidence for their existence showed up in particle accelerators. And antimatter, the fuel of science fiction's future, became science fact in Paul Dirac's mathematical mind before experimentalists noticed antiparticles in cosmic rays.

This behavior certainly lends credence to Tegmark's view above that physical reality *is* mathematics. But that still seems a bit too glib for my tastes, and so furthers the mystery.

Basis for Mathematical Representation of Physical Reality

Returning to Zeno, we see that there is the mathematical issue of the meaning of summing an infinite number of positive quantities to yield a finite result. But there is also the issue of what does the mathematical construct have to do with the physical situation? We are making a lot of assumptions when we form the mathematical representation of the physical problem. We abstract the race course as a 1-dimensional geometric straight line (with no width) — often referred to as the *continuum*. We associate positions on the race course with points on the line. And we associate distances between positions with lengths of line segments between points on the line. So we are

---

[14] Tom Siegfried, "Gravity Waves Exemplify The Power Of Intelligent Equations", February 16, 2016, (https://www.sciencenews.org/blog/context/gravity-waves-exemplify-power-intelligent-equations)

claiming that the mathematical interpretation of the geometric notions of line, points, and lengths is sufficient to explain the physical situation of the race course.

But immediately we are faced with questions about the line or continuum that developed into the late 19th century and early 20th century mathematical field of *point-set topology*. The initial question that arose during the Middle Ages and Renaissance was whether there was a smallest length between points on the line. That is, is the line infinitely divisible into smaller and smaller intervals, or is there a smallest interval? Or more particularly, are there an infinite number of points in a line segment? And if a point has no length, how can a (finite or infinite) sum of points add up to a positive length for the line segment? This became the issue of infinitesimals, which were defined by the contradictory statement that they were infinitely small, positive values. It took a couple of hundred years of mathematical development to clarify the situation, or at least establish a consistent logical foundation.[15]

This issue of mathematical representation of physical reality is extremely complex.[16] In fact, the most difficult part of probability and statistics is not the mathematics but rather its interpretation of physical reality. I see a lot of recommendations that non-math students, who are being taught the most "useful" basics, should be taught a fair amount of probability and statistics, since it arises frequently in our daily lives. But the large number of controversies that develop around any such probability issue indicates that finding a consensus on the interpretation of the math is very difficult, certainly not something that a beginning student should be plagued with. Actually, even in non-probability and statistics situations, such as exponential growth, fluid motion, mechanical statics, etc. how the mathematics should be applied and interpreted for a given physical situation is often the most difficult part of the problem. Part of the reason for this difficulty is the fact that mathematical deductive reasoning is different from physical causal reasoning, as discussed above. So it is often not clear how the two should be married.

So the statement that Zeno's Paradox is resolved *because* an infinite sum can have a finite result is rather problematic and hides a wealth of issues, even to the point of including the issue of what is an "explanation"? But that is a question for another time.

---

[15] A recent take on the problems associated with this mathematical representation of reality, even to the point of resurrecting Zeno, is a series of articles by John Baez on "Struggles with the Continuum", starting 1 Sep 2015 (https://www.physicsforums.com/insights/struggles-continuum-part-1/). As Baez says, "But even if we take a hard-headed practical attitude and leave logic to the logicians, our struggles with the continuum are not over. In fact, the infinitely divisible nature of the real line—the existence of arbitrarily small real numbers—is a serious challenge to almost all of the most widely used theories of physics." And he proceeds to show in the next parts of his series what these challenges are.

[16] For a very basic example, try explaining to an elementary school pupil why an area problem involves multiplication.