Math and Religion

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This was a catchy, misleading title that I could not resist, since my essay is not about math *vs.* religion as one might expect from the title, but rather about math *helping* religion. Back in 2016 I was reading Dr. Bart D. Ehrman's blog[1] that he was writing in preparation for his eventual book, *The Triumph of Christianity*,[2] in which he was considering Rodney Stark's purely mathematical analysis of the growth of Christianity. From Rodney Stark:[3]

> Studies of the rise of Christianity all stress the movement's rapid growth, but rarely are any figures offered. … Nevertheless, we *must quantify*—at least in terms of exploring the arithmetic of the possible—if we are to grasp the magnitude of the phenomenon that is to be explained. … What we need is at least two plausible numbers to provide the basis for extrapolating the probable rate of early Christian growth. Having achieved such a rate and used it to project the number of Christians in various years, we can then test these projections against a variety of historical conclusions and estimates. …
>
> Given our starting number [of 1000 in the year 40 CE], if Christianity grew at the rate of *40 percent per decade,* there would have been 7,530 Christians in the year 100, followed by 217,795 Christians in the year 200 and by 6,299,832 Christians in the year 300 [Table 1.1]. If we cut the rate of growth to 30 percent a decade, by the year 300 there would have been only 917,334 Christians—a figure far below what anyone would accept. On the other hand, if we increase the growth rate to 50 percent a decade, then there would have been 37,876,752 Christians in the year 300—or more than twice von Herding's maximum estimate. Hence 40 percent per decade (or 3.42 percent per year) seems the most plausible estimate of the rate at which Christianity actually grew during the first several centuries.

Table 1.1 Christian Growth Projected at 40 Percent per Decade

Year	*Number of Christians*	*Percent of Population*[a]
40	1,000	0.0017
50	1,400	0.0023
100	7,530	0.0126
150	40,496	0.07
200	217,795	0.36
250	1,171,356	1.9
300	6,299,832	10.5
350	33,882,008	56.5

[a] Based on an estimated population of 60 million.

[1] "The Rate of Christian Growth," 15 May 2016, *The Bart Ehrman Blog*, (http://ehrmanblog.org/the-rate-of-christian-growth/)

[2] *The Triumph of Christianity: How a Forbidden Religion Swept the World*, Simon & Schuster, New York, 2018.

[3] Rodney Stark, "The Arithmetic of Growth," Chapter One in *The Rise of Christianity: How the Obscure, Marginal Jesus Movement Became the Dominant Religious Force in the Western World in a Few Centuries.* Princeton: Princeton University Press, 1996.

Ehrman considered these numbers and some of his own to try to weigh the validity of the estimates. But neither Rodney Stark nor Bart Ehrman described explicitly the underlying mathematical models of exponential growth that they were using and exactly what was meant by a rate of growth. Given the natural audience for the subject, these omissions were not surprising. So I thought I would clarify the math and also offer some variations on the models, which eventually reflected the actual situation more faithfully. I conveyed these ideas and results to Dr. Ehrman and he was able to merge them with the other information he was obtaining.

Exponential Model

So, according to Stark, we have 1000 converts initially in year 40 CE and 6.3 million converts after 260 years. Let $y = C(t)$ represent the number of converts at any moment with t given in years. Let $t = 0$ represent year 40, then $t = 260$ represents year 300. Assume the growth of converts follows a typical *exponential growth law*:

$$dy/dt = ky$$

for some growth constant k yet to be determined. That is, the rate of growth in the quantity *y* is proportional to the amount existing at any moment. Then the solution to the differential equation is[4]

$$y = C(t) = 1000\, e^{kt}.$$

$C(260) = 6.3$ million (number of converts at year 300) implies $k = .033647$[5] or 3.36% converts per year (not quite what Stark got). That is, at the rate of converting a little over 3 people out of every 100 souls each year, Christianity could have grown from 1000 to 6.3 million in 260 years. Note that the formula for k can be written

$$k = (\ln(\text{ending \# converts}) - \ln(\text{starting \# converts})) / \text{duration (years)}$$

% Converts per Year to % Converts per Decade. To convert k (fraction of converts per year) to a fraction per decade, consider

$$[C(t+10) - C(t)]/C(t) = e^{10k} - 1.$$

So if $k = .033647$ per yr, then $e^{10k} = 1.399997$ or a 40% per decade growth, as Stark claimed.

Figure 1 shows an Igor[6] graph that portrays exponential growth rates mentioned by Stark (beginning with 1000 Christians in 40 CE) portrayed in solid lines. The values in Stark's Table 1.1 are shown as black dots. Bart Ehrman proposed his own numbers (20 Christians in 40 CE), which are shown as dashed lines. In order to reach Stark's numbers in 350 CE and a desired 6 million in 312 CE, Ehrman had to assume almost a 60% per decade growth rate, which is extremely high.

Clearly the exponential growth is not sustainable for long, since the number of Christian converts in 370 would exceed Stark's projected 65 million (from 60 million in 300, not shown) for the entire Roman Empire. We need something that takes into account a finite population pool from which the

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4 $dy/dt = ky \Rightarrow \int dy/y = \int k dt \Rightarrow \ln y = kt + \text{const.} \Rightarrow y = y_0 e^{kt}$ where y_0 = the (constant) quantity y at $t = 0$, in this case, $y_0 = 1000$.

5 $C(260) = 6{,}300{,}000 = 1000e^{k260} \Rightarrow \ln 6{,}300 = 260k \Rightarrow k = .033647$.

6 This is a Matlab-like analysis tool that I use for all my mathematical modeling and plots. It is much cheaper than Matlab and provides better graphics. It has interactive on-screen capabilities to carry out and display the results, which are then captured in a quasi-C-like macro language whose fundamental elements are waves or time series. The language can be programmed and compiled to run very fast. I have been using the tool for almost 25 years. Their user support is phenomenal with an ongoing mailing list where more experienced users help the less experienced. I have no relationship with the Wavemetrics company either financial or otherwise. I just find the rare good small software company deserves commendation and support.

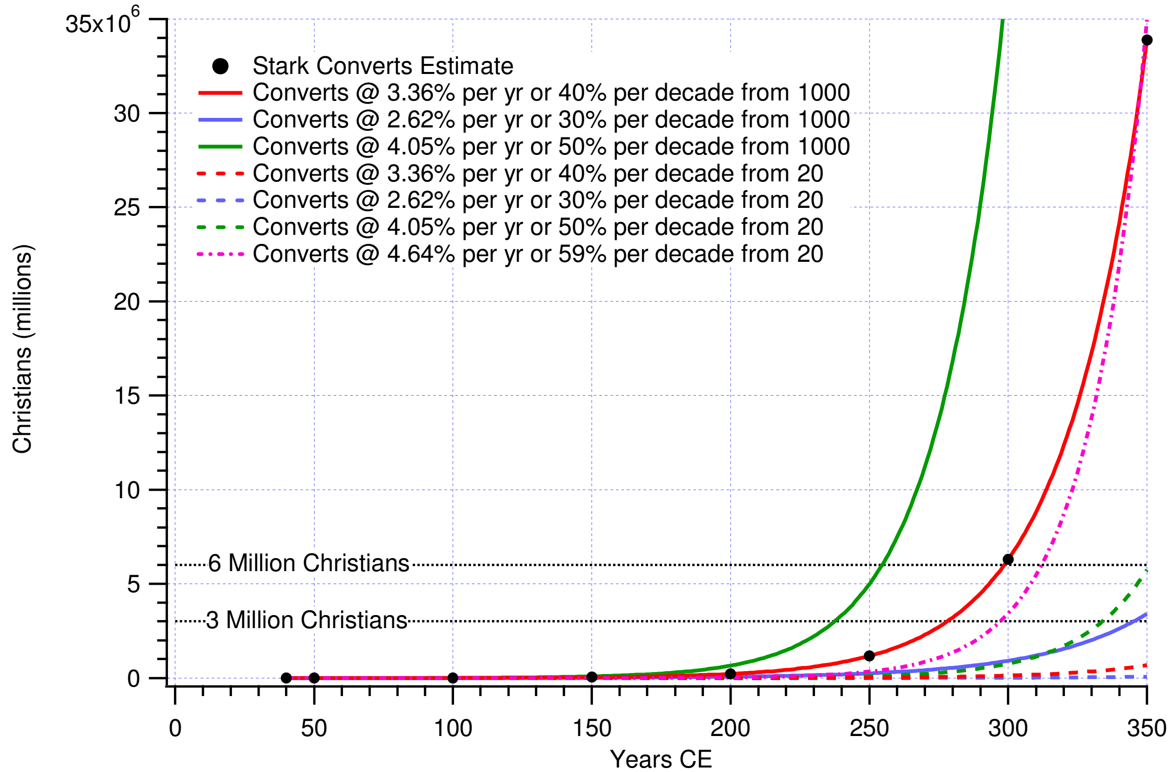

Figure 1 Christian Exponential Growth Rate Comparisons

conversions are taking place. That is, we expect the growth rate to depend not only on the number of existing converts but also on the number of those who are still unconverted. In a finite population, as the number of converts increases, the number of unconverted decreases. So there is an attenuating aspect to this growth rate that would keep the number of converts from exceeding the total population. This formulation is called a *logistic growth model* and is given by

$$\frac{dP}{dt} = rP = (b-d)P \tag{1}$$

$$\frac{dC}{dt} = kC = k_o(1-C/P)C = k_o\left(\frac{P-C}{P}\right)C \tag{2}$$

where the net population growth rate r is given as the difference between the birth rate b and the death rate d for the population P, and where k is the net Christian conversion rate for Christians C, made up of a max constant value k_0 and attenuating factor (1 – C/P) = (P – C)/P for the fraction not converted (where 100 x fraction = %).

Logistic Model

One form of the logistic model employs a constant value for the population $P = P_{max}$. Then equation (2) gives the exact logistic growth model, whose solution[7] is

$$C_t = \frac{P_{max} C_0 e^{k_0(t-t_0)}}{P_{max} + C_0(e^{k_0(t-t_0)} - 1)} \tag{3}$$

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[7] See “Logistic Function”, *Wikipedia*: https://en.wikipedia.org/wiki/Logistic_function, retrieved 5/24/16

where C_0 is the Christian population at time t_0 and k_0 is the growth rate for a pure exponential growth, unattenuated by a finite population P_{max}.

Notice that when t is close to t_0, then $t - t_0$ is close to 0, $e^{k_0(t-t_0)}$ is close to 1, and equation (3) is close to $C_0 e^{k_0(t-t_0)}$ or the exponential growth. Now multiply the numerator and denominator in equation (3) by $e^{-k_0(t-t_0)}$ yielding

$$C_t = \frac{P_{\max} C_0}{P_{\max} e^{-k_0(t-t_0)} + C_0(1 - e^{-k_0(t-t_0)})} \quad (4)$$

So when t becomes large (long time interval), $e^{-k_0(t-t_0)}$ approaches 0, which implies equation (4) approaches the constant value P_{max}.

We use *Wikipedia* estimates for the population of the Roman Empire in years 14 CE and 164 CE given in Table 1.

Table 1 Estimate of the Population of the Roman Empire

Region	Area (1000 km^2)	14 CE Population (millions)	14 CE Density (per km^2)	164 CE Population (millions)	164 CE Density (per km^2)	Population increase (per cent)
Greek East	975.5	20.4	20.9	22.9	23.5	12.3
Greek East (with annexations)	975.5	**20.4**	20.9	**23.1**	23.5	12.3
Latin West	2,364	25.1	10.6	35.7	15.1	42.2
Latin West (with annexations)	2,364	**25.1**	10.6	**38.2**	15.1	42.2
Roman Empire	3,339.5	**45.5**	13.6	**61.4**	15.9	34.9

"Area" includes client kingdoms annexed soon after 14 CE.
After Frier, "Demography", 812, table 5, 814, table 6.
"Demography of the Roman Empire", *Wikipedia*,
(https://en.wikipedia.org/wiki/Demography_of_the_Roman_Empire, retrieved 5/24/16)

In the Igor plot (Figure 2), the red line represents the pure exponential growth of Stark's model, the blue line the pure exponential growth of Ehrman's model (20 converts in 40 CE), and the dashed black line represents the logistic growth for Ehrman's model using the maximum estimated population of the Roman Empire in 400 CE (obtained from an exponential growth model using the population values for 14 CE and 164 CE from Table 1—note this gives a population estimate of 80 million in 300 CE instead of the 60 million assumed by Stark).

The graph shows the logistic growth for Ehrman's model starts out with a high 60% per decade exponential growth, then after 300 CE approximates the slower 40% per decade Stark growth to about 370 CE, at which time the growth of Christians drops off as it approaches the maximum population in 400 CE.

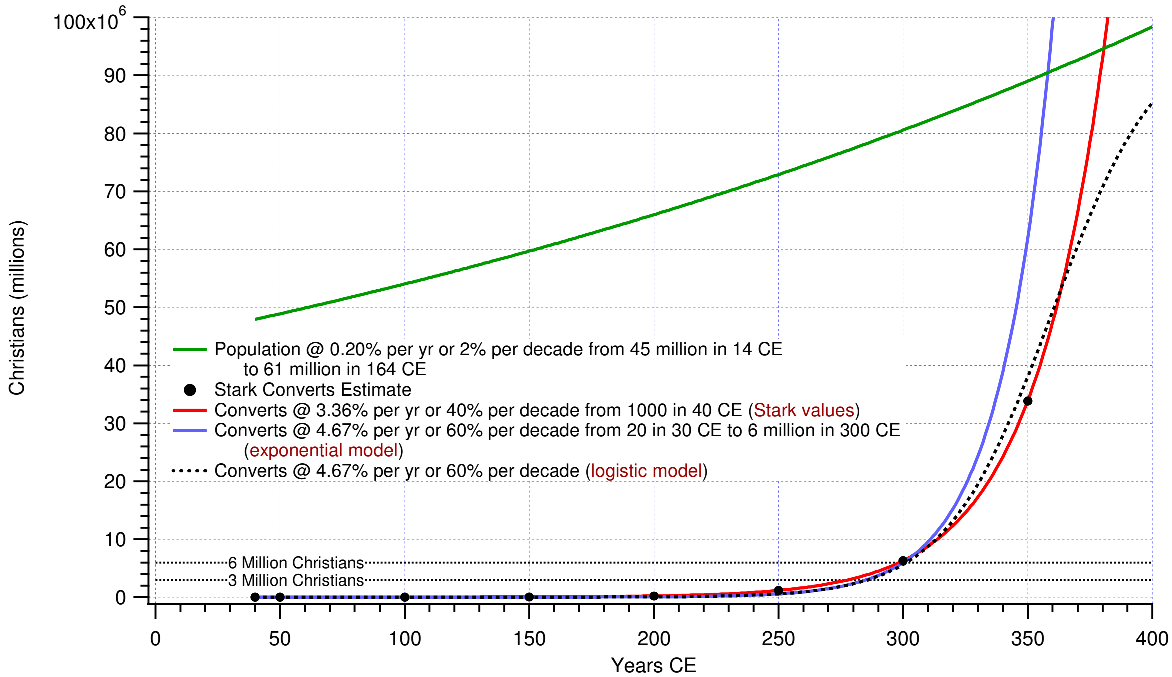

Figure 2 Population and Christian Logistic Growth Plots Including Logistic Model

Iterative Logistic Model

Instead of using a constant population value $P_{\max}$ in equation (2), we will consider using the variable population values P_t from the exponential model in equation (1), which is what we have been plotting already in the green line. The iterative or discrete form of equation (2) for the Christian growth becomes

$$\Delta C = kC\Delta t = k_0(1 - C/P)C\Delta t \text{ , or}$$

$$C_{t+\Delta t} = C_t + \Delta C = C_t\left(1 + k_0(1 - C_t/P_t)\right)\Delta t \qquad (5)$$

We will take Δt to be 1 day. The value k_0 will be the Christian growth rate from the exponential model.

Figure 3 shows the results with the previous logistic model (black dotted curve) using a constant maximum population included for reference. However, we arbitrarily accept Stark's assumption that there were 60 million people in the Roman Empire in 300 CE (and still 45.5 million in 14 CE). As expected, the iterative logistic model (dotted aqua curve) shows a slower growth than the regular logistic model since the early yearly values of the population are less than the maximum in 400 CE.

Ehrman began to modify his estimates, based on early very high conversion rates:[8]

> Thus it appears that the beginning of the Christian movement saw a veritable avalanche of conversions. Possibly many of these are the direct result of the missionary activities of Paul. But there may have been other missionaries like him who were also successful. And so let's simply pick a sensible rate of growth, and say that for the first forty years, up to the time when Paul wrote his last surviving letter, the church grew at a rate of 300%. If the religion started with twenty people in 30 CE, that would mean there were some 1280 by the year 60. That's not at all

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[8] "More on Numbers of Converts," 24 June 2016, *The Bart Ehrman Blog*, (http://ehrmanblog.org/more-on-numbers-of-converts/)

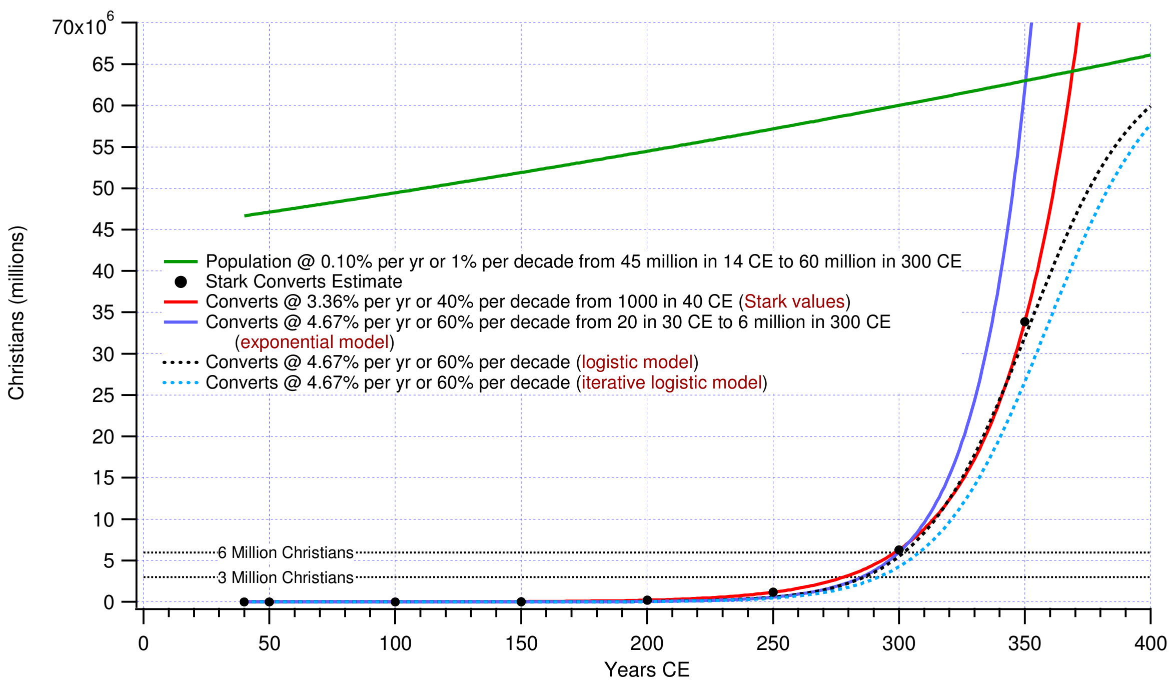

Figure 3 Population and Christian Growth Plots Including Iterative Logistic Model

implausible as a guess. But growth cannot continue at that rate. If it did, a century later, in the year 160, there would be well over a trillion Christians in the world.

So let's say that there was a burst of initial radical enthusiasm generated by the new faith, both among people who had heard Jesus preach during his public ministry and who believed the witnesses who said that he had been raised from the dead and among those evangelized through the extraordinary missionary work of Paul and possibly others like him. After Paul's death there was almost certainly a rapid decline. The change would not be immediate or steady, but we are dealing with ballpark figures here. Say it went down on average to 60% per decade for the next forty years, while there was still a lot of energy and enthusiasm among those who thought not only that Jesus saved them from their sins but that he was coming back very soon, creating a kind of urgency for their message. This would be a rate of growth just under 5% per year. Every year each group of twenty people need to make just one convert. At a rate like that there would then be something like 8381 Christians in the world in the year 100 CE. That sounds about right. …

So based on some educated guessing, we have ballpark figures. Even as such, they are striking. Given the rate adjustments I'm suggesting (in the years 60 CE [from 300% to 60%], 100 CE [to 34%], and 300 CE (to [26%]) here are how the numbers of Christians would break down over time (rounded up to the nearest 1000 starting with 150 CE) [Table 2].

Table 2 Eherman Final Estimates

Year (CE)	Christians: Calculated Estimate	Christians: Spread
30	20	20
60	1280	1,000-1,500
100	8381	7,000-10,000
150	36,000	30,000-40,000
200	156,000	140,000-170,000
250	676,000	600,000-700,000
300	2,920,000	2.5 million-3.5 million
312	3,855,000	3.5 million-4 million
400	29,511,000	25 million-35 million

Figure 4 shows the modifications I made in the models to approximate Ehrman's estimates in Table 2, starting with 8,381 Christians in 100 CE I further assumed that there were 3 million Christians in 300 CE, which would yield about 30 million Christians in 400 CE using the logistic model and about 28 million using the iterated logistic model. Note that the pure exponential model would have yielded 57 million Christians in 400 CE, which is just 10 million short of the total population of 66 million – this seems extreme and supports the use of the logistic models.

Ehrman is effectively introducing the logistic model through his discussion of ever decreasing growth rates in the centuries following the first and his recognition of the diminishing pool of non-Christians to convert. His resulting estimates in the table above are plotted in Figure 4 as red dots. They essentially fall on the logistic model plots.

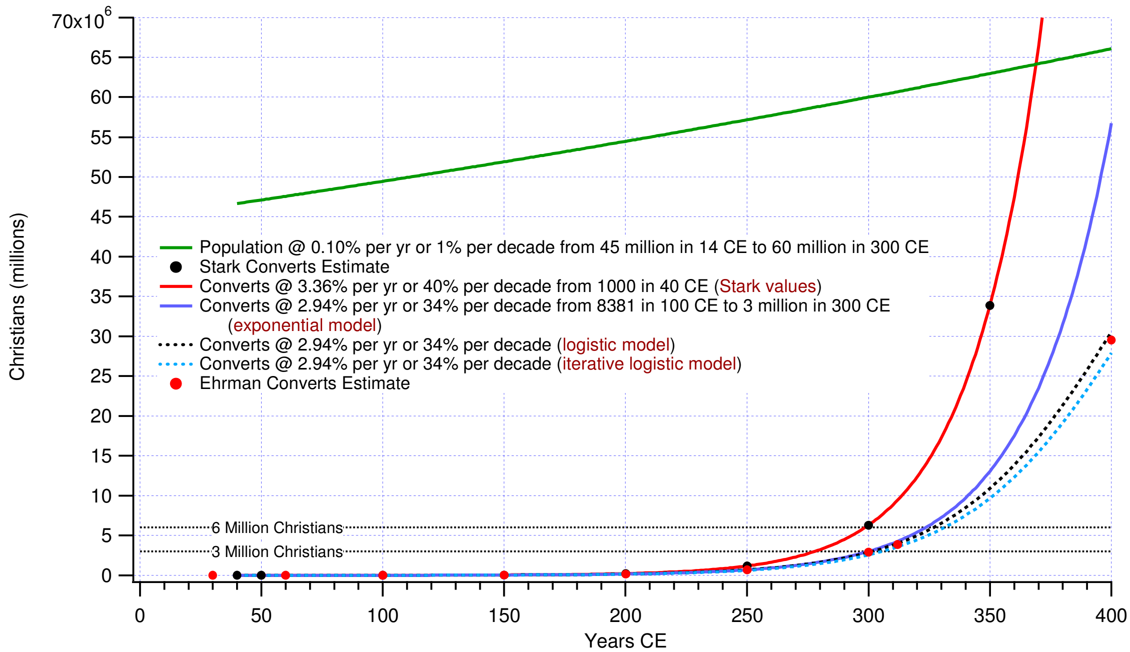

Figure 4 Growth Plots with Ehrman Estimates

As a final detail, Figure 5 shows the diminishing conversion rates in my iterated logistic model. They are a bit higher than those quoted by Ehrman, at least from 300 CE on. I had 33% per decade and Ehrman 26%. I did not reach 26% until 365CE.

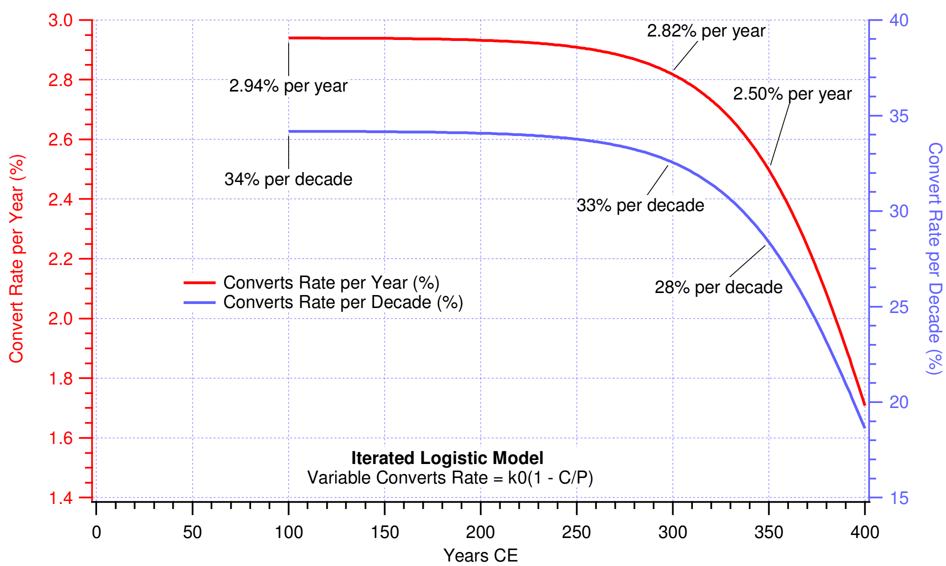

Figure 5 Variable Christian Conversion Rate for the Iterated Logistic Model

I will give Bart Ehrman the last word:[9]

> I need to stress that we are not talking about implausible rates of growth, even though the numbers at the end of the period are staggering. For the fourth century, if the rate is 26% per decade, that is just over 2.34% per year. Every hundred Christians, among them, need to convert just two or three people. And the conversions include everyone who begins to adopt Christian practices. If the head of a household converts, and he brings his wife and three children into the fold so that they too adopt the new faith (as happens in Mormonism as well as in fourth century Christianity), then you have five new members. We know that these kinds of "family conversions" occurred from the very beginning of the Christian movement. … It would thus be a relatively simple matter for the church to grow at a rate of 2.5% per year – the rough rate of growth we are hypothesizing for the fourth century. It would simply mean that every group of a hundred Christians would have to witness one male-adult convert, along with his family, every two years.

[9] "More on Numbers of Converts," 24 June 2016, *The Bart Ehrman Blog*, (http://ehrmanblog.org/more-on-numbers-of-converts/)