Geometric Puzzle Medley

(16 August 2018)

Jim Stevenson

This is a collection of simple but elegant puzzles, mostly from a British high school math teacher Catriona Shearer @Cshearer41 (https://twitter.com/Cshearer41), for which I thought I would show solutions (solutions for a number of them had not been posted yet on Twitter).

Four Squares Puzzles

The Puzzle 1 (Figure 1) showed up on 8 August 2018 from Catriona Shearer @Cshearer41 (https://twitter.com/Cshearer41/status/1027129834311438337).

This puzzle was followed up on 9 August 2018 with two variations (Figure 2 and Figure 3) from @Five_Triangles

https://twitter.com/Five_Triangles/status/1027638888393781249

and

https://twitter.com/Five_Triangles/status/1027727386920538112, respectively)

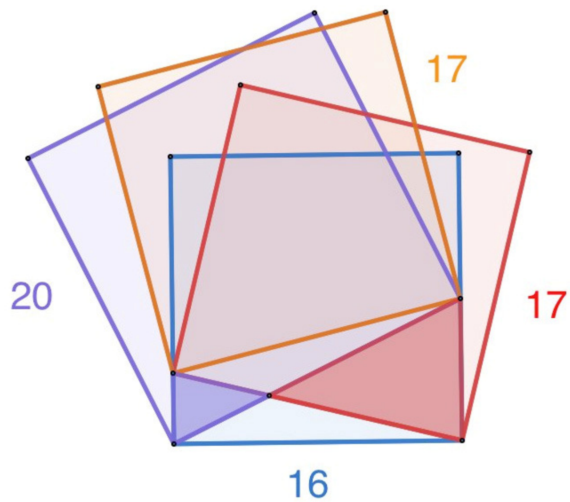


Figure 1 Puzzle 1. The numbers show the areas of the overlapping squares. What's the difference between the areas of the red and purple triangles?

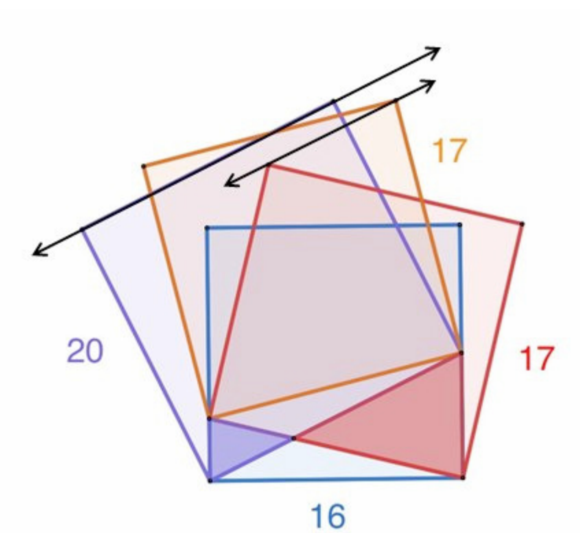


Figure 2 Puzzle 2. Prove the side of the square and the line through the 2 vertices are parallel.

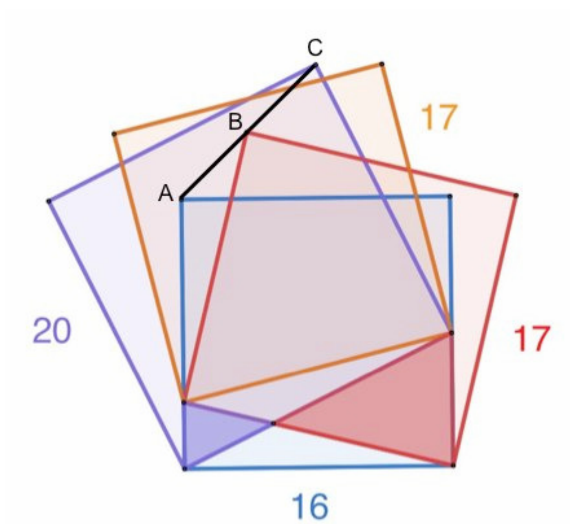


Figure 3 Puzzle 3. A clever (lazy?) teacher could milk this diagram for a week's worth of problems. Show points A, B and C are collinear and find AB:BC.

I saw Puzzles 3 and 4 before I saw the original Puzzle 1, so I misinterpreted the numbers to be the lengths of the sides of the squares rather than the areas (others seemed to have made the same mistake). This led to a lengthy discussion of why the puzzles were erroneous. But now that I see the numbers are areas, I also see the puzzles are correct and I worked out the solutions (which have not been posted yet on the referenced twitter accounts that I could see at the time of writing).

Puzzle 1

I have relabeled the diagram in Figure 1 with the lengths of the edges (Figure 4). The (vertical) base of the blue triangle x turns out to be 1 from an application of the Pythgorean Theorem: $x^2 = (\sqrt{17})^2 - 4^2 = 1$. From the geometry we can conclude that the (vertical) base y of the red triangle is 2x, or 2. Or we could apply the Pythagorean Theorem again to get $y^2 = (\sqrt{20})^2 - 4^2 = 4$, so that again y = 2.

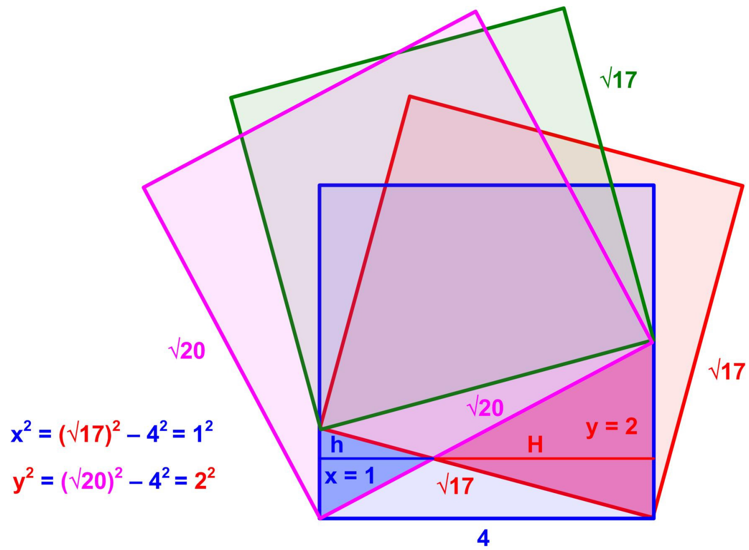


Figure 4

The next task is to find the altitudes h, H of the blue and red triangles respectively. We know h + H = 4. Furthermore, the two triangles are similar, since they have the same angles. The scale factor of red to blue is y : x = 2. Therefore, H = 2h as well. So 4 = 3h or h = 4/3. Then H = 4 – 4/3 = 8/3. Therefore, the area of the blue triangle = ½ xh = ½ 1·4/3 = 2/3, and the area of the red triangle = ½ yH = ½ 2·8/3 = 8/3. So the difference is 8/3 – 2/3 = 2.

Puzzle 2

Figure 2 has been filled in with the values of x = 1 and y = 2x = 2 from Puzzle 1 (Figure 5). The slope of the large pink square is obtained from the right triangle in the lower right corner of the blue square, and is 2x / 4 = 1/2.

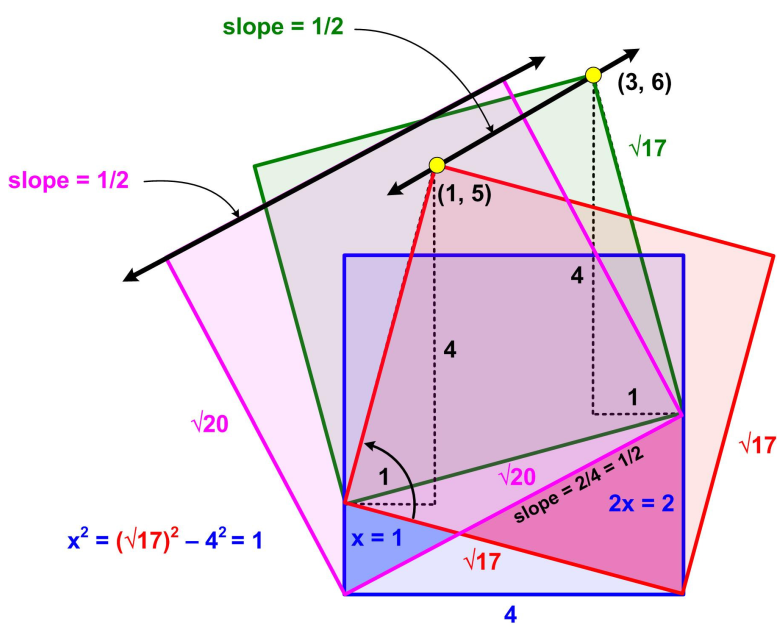


Figure 5

To compute the slope of the line joining the two corners of the red and green squares we need to find the coordinates of the two points. First, rotate counter clockwise 90° the 4-1-√17 right triangle in the lower left corner of the blue square. From this we determine that the coordinates of the corner of the red square are (1, 5). Flip the dashed triangle and move it to the lower right corner of the green triangle. From this we compute the coordinates of the upper right corner to be (3, 6). Then the slope of the line through these two points is (6 – 5)/(3 – 1) = 1/2, which is the same as the previous slope. Therefore, the two lines are parallel.

Puzzle 3

We have added the results of the previous figures to the original Figure 3 (see Figure 6). Since point C is the upper corner of the pink square rather than the green square, we compute its coordinates differently. Namely, we rotate clockwise 90° the 2-4-√20 right triangle in the lower right corner of the blue square. From this we compute the coordinates of point C to be (2, 6). Then the slope of the line BC is (6 – 5)/(2 – 1) = 1.

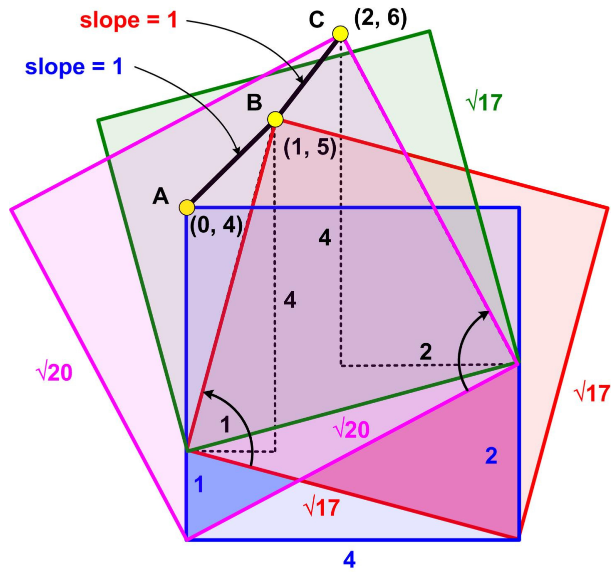


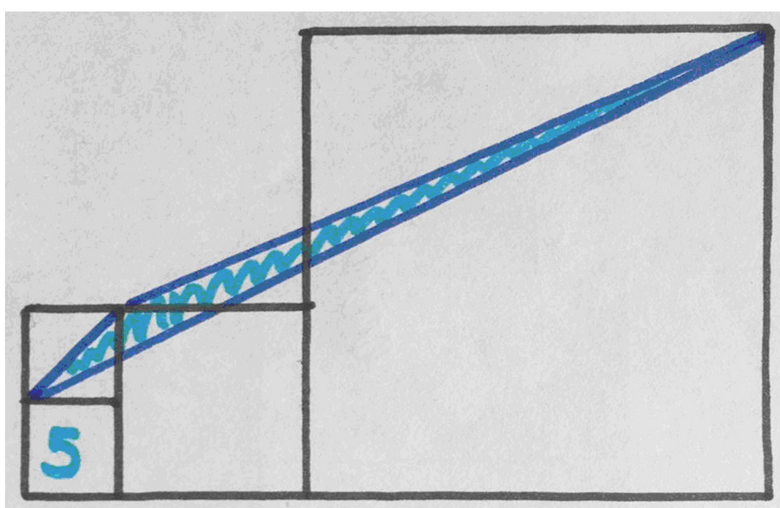
Figure 6

The coordinates of point A are trivially (0, 4), so that the slope of line AB is (5 – 4)/(1 – 0) = 1. So AB || BC and have a point in common. Therefore, the points A, B, and C are collinear.

Now $|AB|^2 = (1-0)^2 + (5-4)^2 = 2$, so $|AB| = \sqrt{2}$. $|BC|^2 = (2-1)^2 + (6-5)^2 = 2$ also, so $|BC| = \sqrt{2}$ as well. Therefore, AB : BC as √2 : √2 = 1.

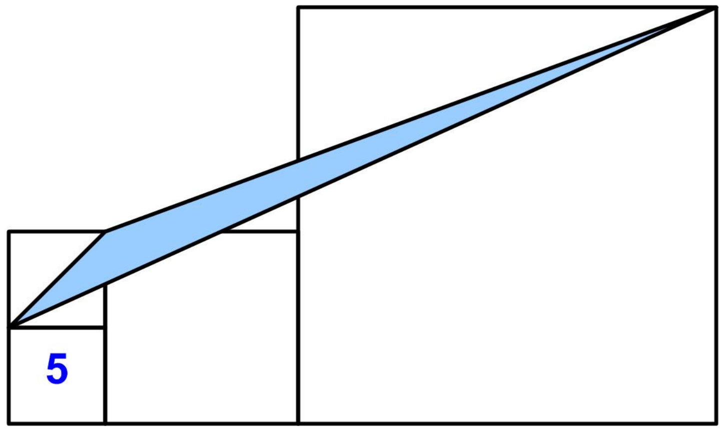
Five Problem

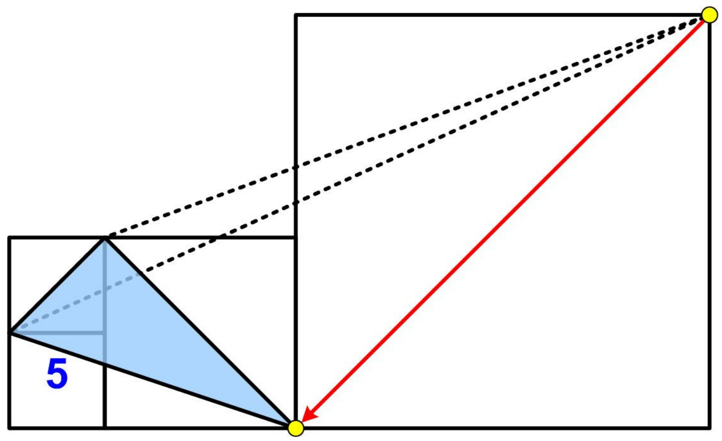
This elegant problem was posed by Catriona Shearer on Twitter on 10 August 2018 (https://twitter.com/Cshearer41/status/1027844515338616832). It turns out not to be hard, but the original impression is that there is not enough information. So when the solution dawns, it is most gratifying. The commenters on the problem had a similar reaction. Here is Catriona's statement:

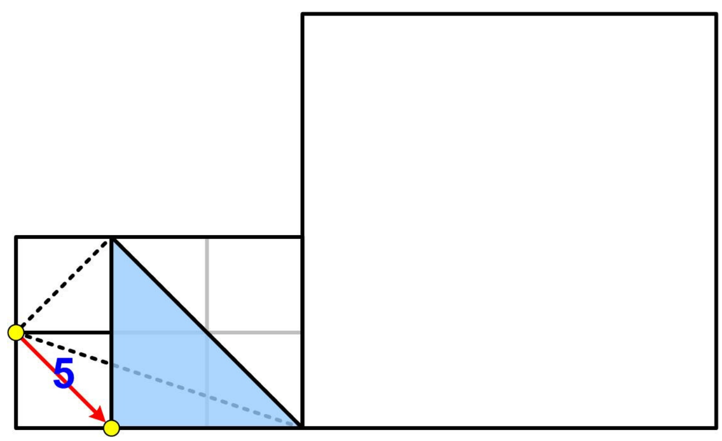


The area of the bottom left square is 5. What's the area of the blue triangle?

The solution is practically wordless:







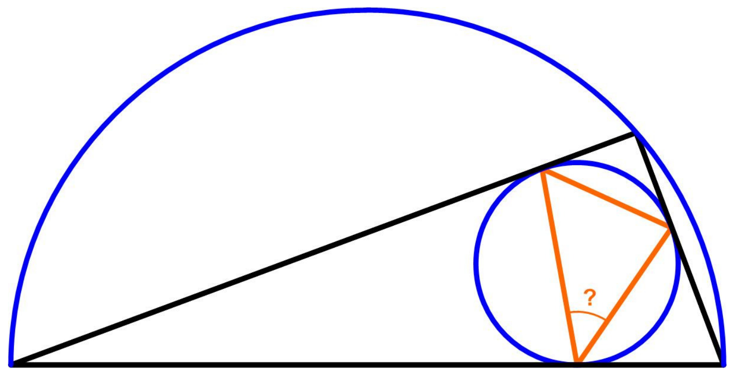
Sheering a triangle by moving its top vertex parallel to its base does not change its area, since the altitude of the triangle remains constant. This is a powerful idea and is used heavily by Newton in his proof that the line from the sun to the planets sweeps out equal areas in equal times as the planet orbits the sun. One Twitter commenter noted the medium square equals four of the smallest squares, and so the area of the triangle being half the medium square means it is 10 units. So no explicit calculations had to be done. Very elegant.

Something noted by Catriona Shearer in her comments was that the area of the blue triangle is independent of the size of the largest square! Amazing.

Circle-Triangle Puzzles

Catriona Shearer @Cshearer41 (https://twitter.com/Cshearer41) has supplied some more easy but elegant geometric puzzles:

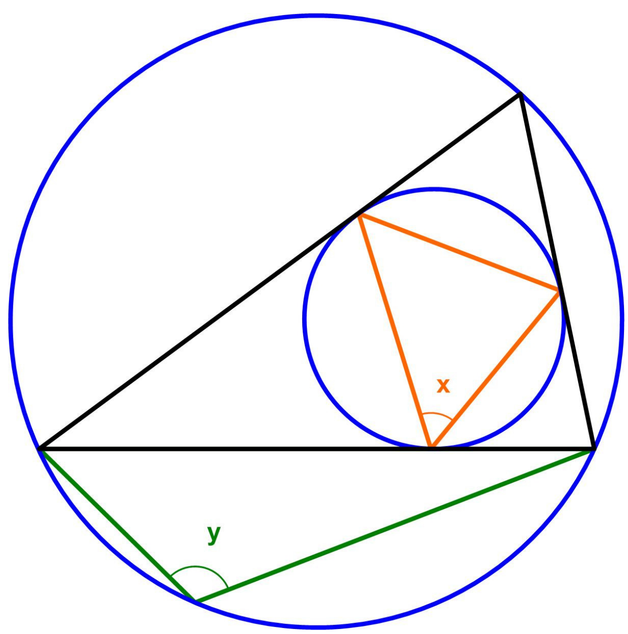
Puzzle 1



A triangle in a circle in a triangle in a semicircle... what's the angle?

(https://twitter.com/Cshearer41/status/1029760468406157313) 15 August 2018 09:03 AM PT)

Puzzle 2



Nice! The original picture was a semicircle, but if we remove that constraint, this would be a nice follow-up question: find y in terms of x.

(https://twitter.com/Cshearer41/status/1030032257153724419, 16 August 2018 3:03 AM PT)

Puzzle 1 Solution

After I solved this a different way, the answer suggested there must be a simpler approach, and there was.

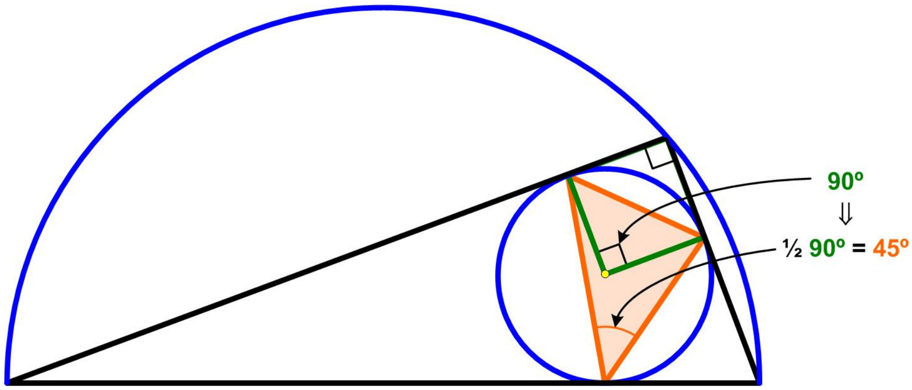


Figure 7

First we notice the black triangle inscribed in a semicircle is a right triangle. (It is one half the central angle that spans the same arc of the circle and therefore one half of 180°.) Figure 7 shows the addition of the radii of the small blue circle that are drawn to the tangent points of the circle and black triangle. Since they are perpendicular to the sides of the triangle, we have a quadrilateral with 3 angles summing to 270°, so that last angle must also be 90°. Since the unknown angle is one half its corresponding central angle, it must be 45°.

Puzzle 2 Solution

First draw a line from the vertex A of the black triangle to the center C of the circle. Then draw the radii CB and CD to the tangent points of the circle and black triangle. Since radii are perpendicular to tangents to the circle, ABC and ADC are right triangles. With a hypotenuse in common and identical radial legs the two triangles are congruent. Therefore their angles are identical as shown.

Then we have:

$$\alpha + \beta = 90$$

$$x = (1/2)\ 2\beta = \beta$$

$$y + 2\alpha = 360/2 \Rightarrow \alpha = 90 - y/2$$

$$\Rightarrow \quad 90 = \alpha + \beta = (90 - y/2) + x$$

$$\Rightarrow \quad y = 2x$$

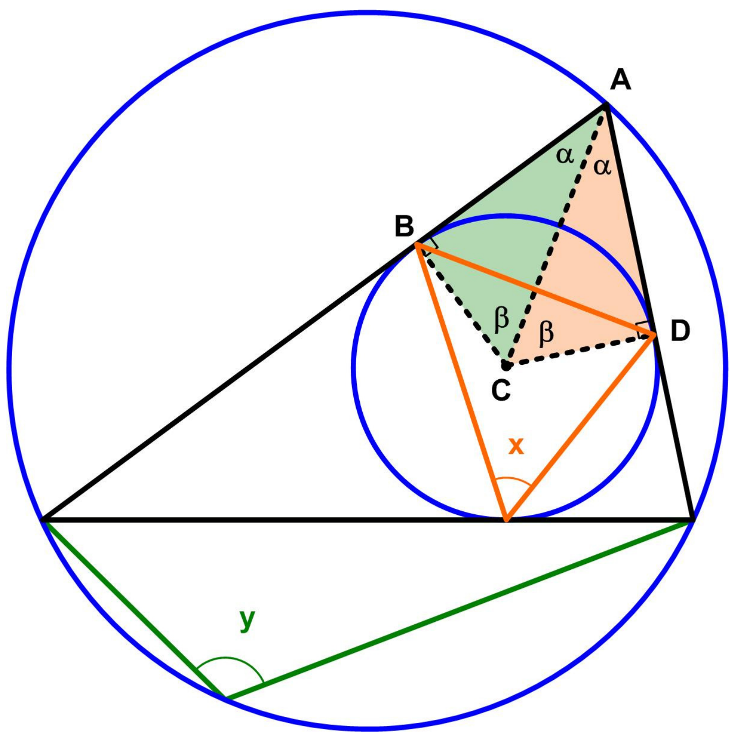


Figure 8

Clearly this general result can also be used to solve Puzzle 1, since the base of the black triangle being a diameter of the large circle means y = 90°. Therefore x = y/2 = 45°.

Circle-Area Puzzle

I came across another simple but elegant geometric puzzle submitted by Catriona Shearer on Twitter on 12 August 2018 (https://twitter.com/Cshearer41/status/1028561011299807232):

The two squares are identical, and the pink line is a diameter of the circle [centered on the square corners]. What's the area of the pink triangle?

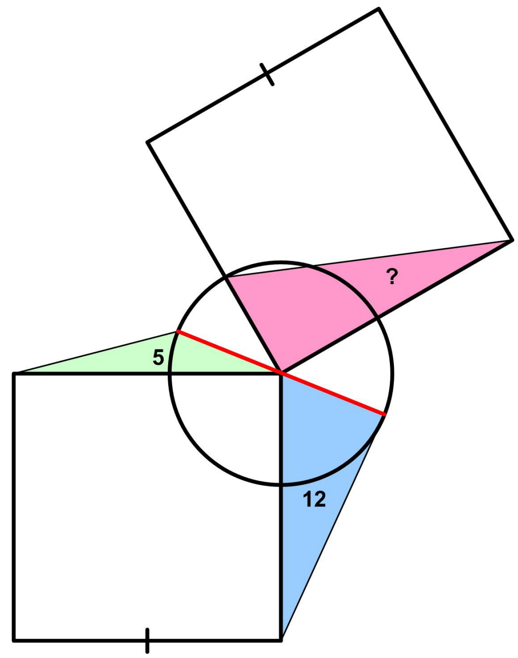


Figure 9

Solution

First, designate the edge of both squares x and the radius of the circle r. Then the unknown area becomes A = ½ x r. Since this area does not change if its square is pivoted around the center of the circle, the first thing we do is rotate it until its lower edge is in line with the upper edge of the other square (see Figure 10). This means the angles in the blue and green triangles at the center of the circle become complementary, that is, they add to 90°. If we call the angle in the green triangle α and the one in the blue triangle β, then we can compute the altitudes and areas of both these triangles as

Green: $5 = \frac{1}{2} x r \sin\alpha = A \sin\alpha$

Red: $12 = \frac{1}{2} x r \sin\beta = A \sin(90° - \alpha) = A \cos\alpha$

This immediately suggests the way to get rid of the trigonometric functions is to square both sides and add the two equations. Since $\cos^2\alpha + \sin^2\alpha = 1$, We have $A^2 = 5^2 + 12^2 = 13^2$. So A = 13.

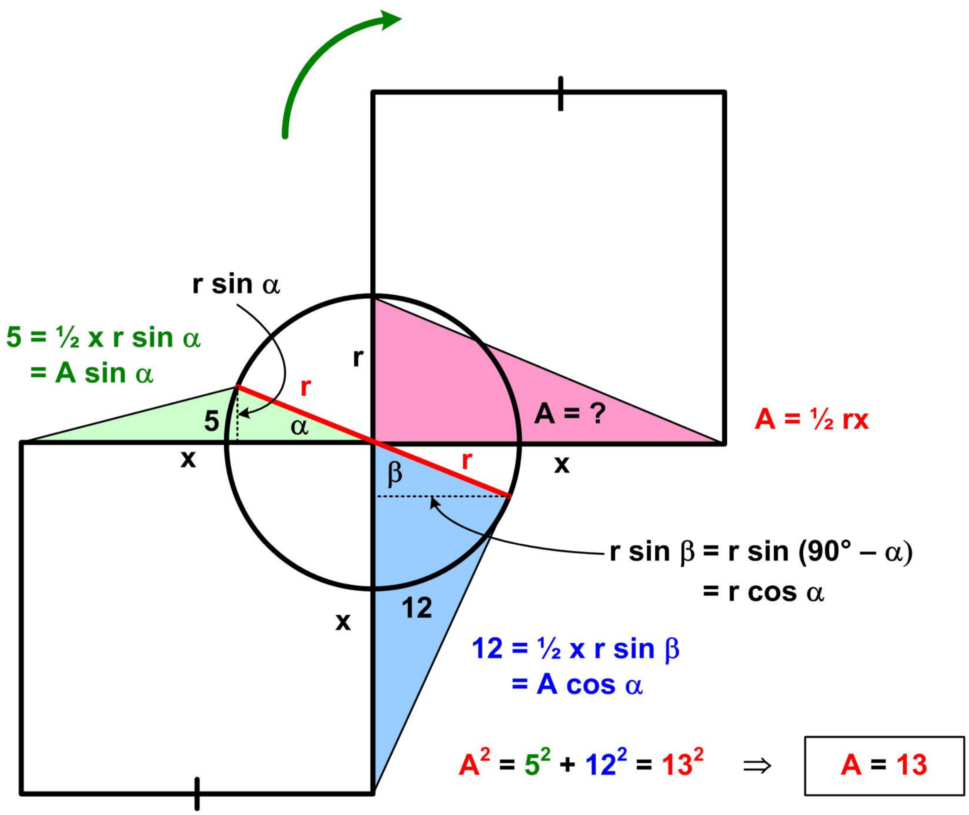

r sin α
5 = ½ x r sin α
= A sin α
r
r
5
α
A = ?
A = ½ rx
x
r
x
β
r sin β = r sin (90° – α)
= r cos α
12
x
12 = ½ x r sin β
= A cos α
A² = 5² + 12² = 13²
⇒
A = 13

Figure 10
