The Four Travelers Problem

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Another problem from the Futility Closet site:

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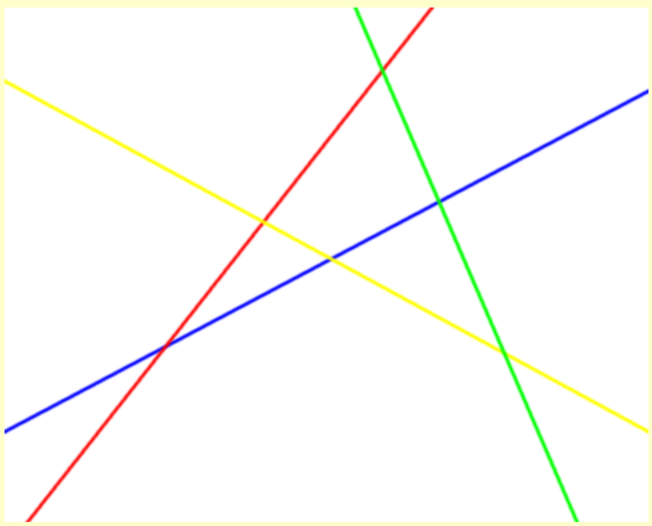
(http://www.futilitycloset.com/2016/05/26/four-travelers-problem/, retrieved 5/30/16)



Four straight roads cross a plain. No two are parallel, and no three meet in a point. On each road is a traveler who moves at some constant speed. If Blue and Red meet each other at their crossroad, and each of them meets Yellow and Green at their respective crossroads, will Yellow and Green necessarily meet at their own crossroad?

Futility Closet Solution

Surprisingly, the answer is yes. Rob Fatland offers a beautiful solution at the CTK Exchange:[1]

> Imagine the whole scenario unfolding from Blue's rest frame; that is, regard Blue as unmoving and consider the movements of the other travelers relative to him. What does he see? Objectively we know that each traveler moves along a straight line at a constant speed, eventually encounters Blue, and moves on, so from Blue's perspective each of them moves directly toward him on a straight line, passes through his position, and continues.
>
> Very well, let Red do that. But we know that Red also encounters Yellow and Green

[1] http://www.cut-the-knot.org/htdocs/dcforum/DCForumID4/680.shtml [JOS: I was not able to find the cited quote at the reference given.]

without deviating from his own path. And we know that (from Blue's perspective) Yellow and Green are also traveling straight lines that intersect Blue's position. This can only mean that Red, Yellow, and Green are all traveling along the same straight line from Blue's point of view. And this means that Yellow and Green must meet one another.

(It might be objected that two points traveling the same line needn't meet if they're going in the same direction at the same speed. But here this would mean that two of the roads are parallel, and that possibility is excluded by the conditions of the problem.)

05/27/2016 UPDATE: Reader Derek Christie recalls a three-dimensional solution: "Make a time axis perpendicular to the plane. Then each traveller moves in a straight line trajectory through this 3D space. Blue and Red meet at a particular place and time, so their two trajectories must meet at a point, and these two trajectories define a plane. Both Yellow and Green trajectories meet the Blue and Red trajectories and so also lie in this same plane. So Yellow and Green must also meet somewhere in that plane."[2]

My Solution

I was not able to understand the solution given at first, so I tried to solve the problem on my own. Once I did, I was able to see what the Futility Closet solution was getting at. Certainly diagrams were needed to make sense of it all, and that is what I provided.

First, we need to notice a geometric property which is a consequence of the constant speed, straight line paths, which is illustrated in Figure 1. Let $\mathbf{r_B}$ and $\mathbf{r_R}$ be the velocity vectors for the

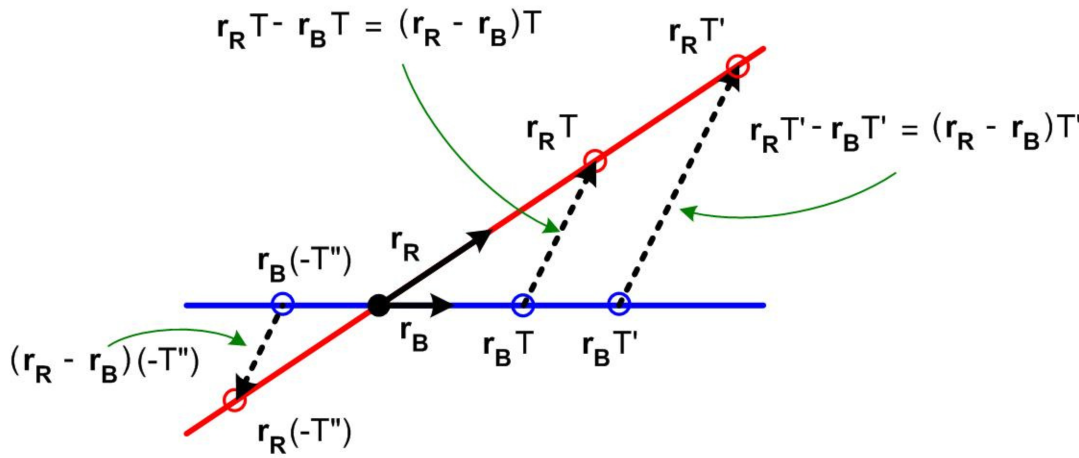


Figure 1 Parallel Time Lines

travelers along the Blue and Red lines respectively. If T represents an interval of time, then $\mathbf{r_B}T$ and $\mathbf{r_R}T$ represent distances traveled along the Blue and Red lines respectively in time T, starting at their common intersection point represented by the black dot. The separation of the end points of the distances traveled is given by the vector $\mathbf{r_B}T - \mathbf{r_R}T = (\mathbf{r_B} - \mathbf{r_R})T$, shown as a dashed vector in Figure 1. For any other time interval T', the distance separation is given by a scalar times the fixed vector $\mathbf{r_B} - \mathbf{r_R}$, that is $(\mathbf{r_B} - \mathbf{r_R})T'$. So all these **separation vectors are parallel** and represent **constant time lines**. Consequently the dashed separation segments and the Red and Blue segments make up similar triangles.

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[2] JOS: Seeing how these intersecting lines also intersect *at the same time* is more easily seen in my discussion below that essentially portrays the inclined plane as a series of parallel constant time contours.

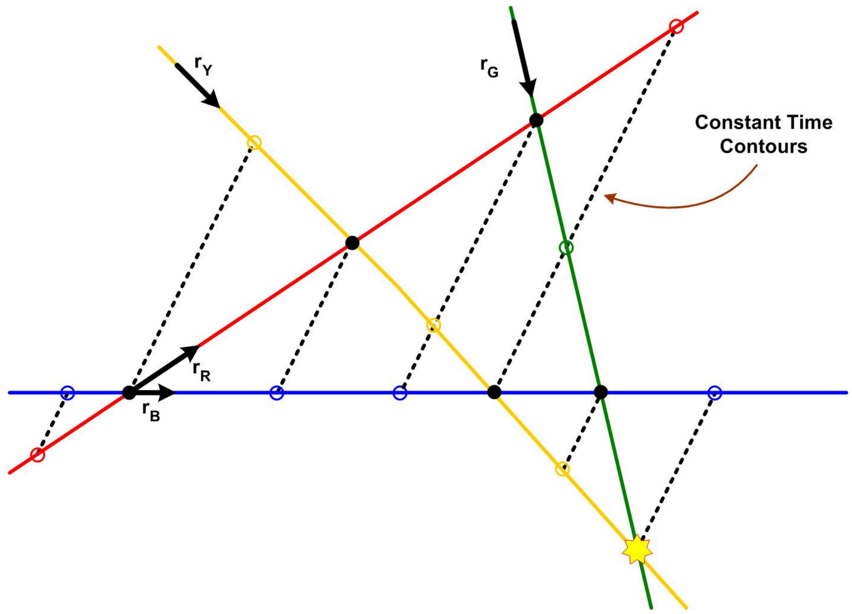

Figure 2 The Four Travelers with Parallel Time Lines Showing Separation Segments

This means we can represent the four travelers along their respective roads as shown in Figure 2 with the parallel (dashed) lines of separation intersecting the colored paths at the same times. It is easy to see the separation between the Yellow and Green travelers decreases until it vanishes at the intersection of the two paths with their simultaneous arrival.

These (dashed) lines must be the lines referred to in the Futility Closet solution where the Blue traveler sees the other travelers each moving towards him along a line as he progresses along the Blue path until they meet and then separate.

One detail should be made clear. The speed relationships of the Yellow and Green travelers along their paths with respect to the Blue traveler along his path are shown in Figure 3. As with the Red and Blue travelers, the relationships are defined by similar triangles with the defining triangle indicated in bold. Besides the points of intersection with the Blue path, these triangles are dictated by the intersections of their respective paths with the Red line, and the corresponding times on the Blue path at which these intersections occur.

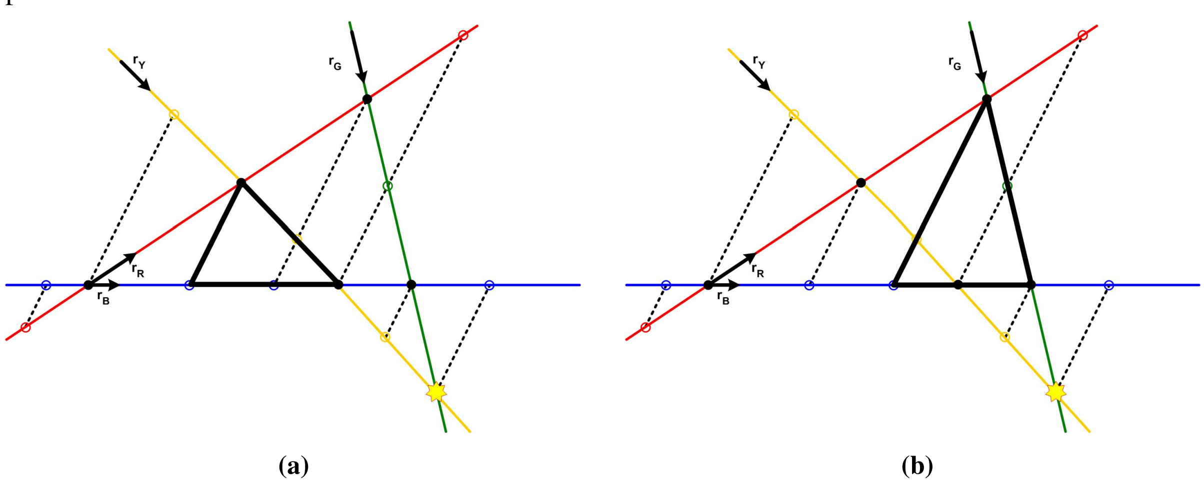

Figure 3 Speed Relationships of Yellow and Green Lines With Respect To the Blue Line