The Diluted Wine Puzzle

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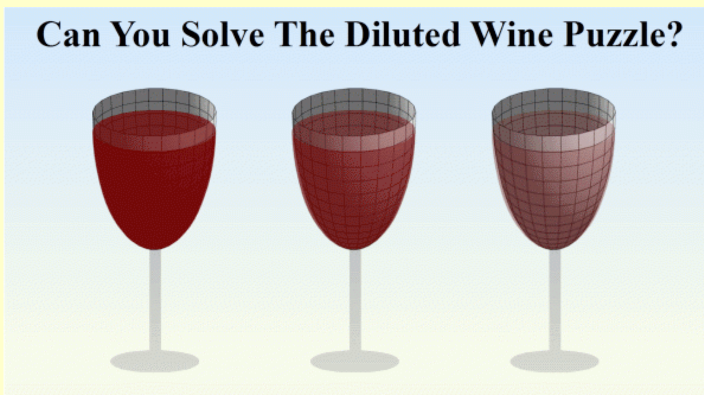
This was a rather intricate puzzle from Presh Talwalkar. I found his solution a bit hard to follow, so I tried for a clearer presentation.

(https://mindyourdecisions.com/blog/2016/11/27/can-you-solve-the-diluted-wine-puzzle-famous-16th-century-math-problem/, retrieved 9/9/2017)

Can You Solve The Diluted Wine Puzzle?

Presh Talwalkar, November 27, 2016

A servant has a method to steal wine. He removes 3 cups from a barrel of wine and replaces it with 3 cups of water. The next day he wants more wine, so he does the same thing: he removes 3 cups from the same barrel (now with diluted wine) and replaces it with 3 cups of water. The following day he repeats this one more time, so he has drawn 3 times from the same barrel and has poured back 9 cups of water. At this point the barrel is 50% wine and 50% water. How many cups of wine were originally in the barrel? Watch the video for a solution.



Talwalkar Answer To The Diluted Wine Puzzle

Suppose the barrel started with x cups of wine. We will keep track of how much wine is in the barrel and the concentration of wine in the barrel. Initially the barrel has x cups of wine and it is 100% wine (a concentration of 1).

Wine Amount (starting),	Wine Concentration (starting)
x,	1

The servant first takes 3 cups of wine and replaces them with water. The amount of wine left is $x - 3$, and the concentration is the amount of wine, $x - 3$, divided by the total volume of liquid, which is x.

Wine Amount (step 1),	Wine Concentration (step 1)
$x - 3$,	$(x - 3)/x$
$x - 3$,	$1 - 3/x$

The servant then removes 3 cups from the barrel, and each cup contains a concentration $1 - 3/x$ of wine. The amount of wine left is $x - 3 - 3(1 - 3/x)/x$, and the concentration is the amount of wine, $x - 3 - 3(1 - 3/x)/x$, divided by the total volume of liquid, which is x.

Wine Amount (step 2), Wine Concentration (step 2)

$$x - 3 - 3(1 - 3/x), \quad (x - 3 - 3(x - 3)/x)/x$$

$$x - 6 + 9/x, \quad (x - 6 + 9/x)/x$$

$$x - 6 + 9/x, \quad (1 - 3/x)^2$$

The servant finally removes 3 more cups from the barrel, and each cup contains a concentration $(1 - 3/x)^2$ of wine. The amount of wine left is $x - 6 + 9/x - 3(1 - 3/x)^2$, and the concentration is the amount of wine divided by the total volume of liquid, which is x.

Wine Amount (step 3), Wine Concentration (step 3)

$$x - 6 + 9/x - 3(1 - 3/x)^2, \quad (x - 6 + 9/x - 3(1 - 3/x)^2)/x$$

$$x - 9 + 27/x - 27/x^2, \quad (x - 9 + 27/x - 27/x^2)/x$$

$$x - 9 + 27/x - 27/x^2, \quad (1 - 3/x)^3$$

The final concentration should be equal to 50%, or 1/2.

$$(1 - 3/x)^3 = 1/2$$

$$(1 - 3/x) = 1/2^{1/3}$$

$$2^{1/3}x - 3(2^{1/3}) = x$$

$$x(2^{1/3} - 1) = 3(2^{1/3})$$

$$x = 3(2^{1/3})/(2^{1/3} - 1) \approx 14.54$$

Thus the original barrel contained approximately 14.5 cups of wine.

There is a shortcut to solving this problem! You can save many steps by noticing the concentration is $1 - 3/x$ after the first step.

Wine Amount (step 1), Wine Concentration (step 1)

$$x - 3, \ 1 - 3/x$$

The subsequent steps iterate the same process of removing 3 cups and then diluting the wine with water. Accordingly, the wine is diluted by the same percentage in each step. To find the new concentration, multiply by the factor $1 - 3/x$.

Wine Concentration (step 2)

$$(1 - 3/x)^2$$

Wine Concentration (step 3)

$$(1 - 3/x)^3$$

Now we can set the concentration equal to 1/2 and find the answer as before.

Sources

- I read this problem in Famous Puzzles of Great Mathematicians (https://amzn.to/2eGt3UQ) by Miodrag S. Petkovic. The puzzle appears in Niccolo Tartaglia's work "General Trattato di Numeri" (1556).
- StackExchange has the idea to multiply concentrations
http://puzzling.stackexchange.com/questions/28776/turning-wine-into-water/28781

My Solution

I think the key to the problem is to consider the 3 cups of liquid removed from the barrel in terms of a fraction of the liquid in the barrel, namely, ${}^{3}/_{V}$ of the volume V of liquid. Then for any uniformly well-mixed subset F of the volume (such as the wine), the fraction ${}^{3}/_{V}$ of F will be removed in the 3 cups of liquid (or equivalently, ${}^{F}/_{V}$ represents the fraction of the 3 cups consisting of F). This means arithmetically if W is the amount of wine at each step and H the amount of water, then

$$V - 3 = V - ({}^{3}/_{V})V = (W + H) - ({}^{3}/_{V})(W + H) = [W - ({}^{3}/_{V})W] + [H - ({}^{3}/_{V})H]$$

The following table shows the sequence of steps and what happens to the wine and the water in the barrel after each step.

Water In	Wine	Water	Liquid Out
	V	0	
	↓	↓	→ **3** cups = $({}^{3}/_{V})V$
	$V - 3$	0	
	$= (1-{}^{3}/_{V})V$	0	
3 cups →	↓	↓	
	$W_1 = (1-{}^{3}/_{V})V$	$H_1 = 3$	
	↓	↓	→ **3** cups = $({}^{3}/_{V})V$
	$W_1 - ({}^{3}/_{V})\, W_1$	$H_1 - ({}^{3}/_{V})\, H_1$	
	$= (1-{}^{3}/_{V})\, W_1$	$= (1-{}^{3}/_{V})\, H_1$	
3 cups →	↓	↓	
	$W_2 = (1-{}^{3}/_{V})\, W_1$	$H_2 = 3 + (1-{}^{3}/_{V})\, H_1$	
	↓	↓	→ **3** cups = $({}^{3}/_{V})V$
	$W_2 - ({}^{3}/_{V})\, W_2$	$H_2 - ({}^{3}/_{V})\, H_2$	
3 cups →	↓	↓	
	$W_3 = (1 - {}^{3}/_{V})\, W_2$	$H_3 = 3 + (1 - {}^{3}/_{V})\, H_2$	
	$= (1 - {}^{3}/_{V})^2\, W_1$	$= 3 + (3 + 3(1-{}^{3}/_{V}))(1-{}^{3}/_{V})$	
	$= (1-{}^{3}/_{V})^3\, V$	$= 3(1+(1-{}^{3}/_{V}) + (1-{}^{3}/_{V})^2)$	

Then 50% of the liquid at the end being wine implies that half the liquid is wine, that is,

$$(1-{}^{3}/_{V})^3\, V = \tfrac{1}{2}V \quad \text{or} \quad (1-{}^{3}/_{V})^3 = \tfrac{1}{2}$$

Therefore the original amount of wine V is

$$V = \frac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$$

As you can see, I avoid multiplying things out during the intermediate steps, contrary to Presh Talwalkar. It is easier to see patterns and to avoid arithmetic mistakes.