The Barrel of Beer

29 January 2019

Jim Stevenson

This is a great puzzle by H. E. Dudeney involving a very useful technique.

76.—THE BARREL OF BEER. ([1] p.14)

A man bought an odd lot of wine in barrels and one barrel containing beer. These are shown in the illustration, marked with the number of gallons that each barrel contained. He sold a quantity of the wine to one man and twice the quantity to another, but kept the beer to himself. The puzzle is to point out which barrel contains beer. Can you say which one it is? Of course, the man sold the barrels just as he bought them, without manipulating in any way the contents.

Solution

The nifty solution to this problem requires some preliminaries that are useful for a number of problems of this nature.

Preamble: Digital Roots and Modular Arithmetic

From *Wikipedia* (https://en.wikipedia.org/wiki/Digital_root), "The **digital root**[1] (also **repeated digital sum**) of a non-negative integer is the (single digit) value obtained by an iterative process of summing digits, on each iteration using the result from the previous iteration to compute a digit sum. The process continues until a single-digit number is reached. For example, the digital root of 65,536 is 7, because 6 + 5 + 5 + 3 + 6 = 25 and 2 + 5 = 7. Digital roots can be calculated with congruences in modular arithmetic rather than by adding up all the digits, a procedure that can save time in the case of very large numbers."

Modular Arithmetic

"Modular arithmetic" refers to adding and multiplying integers modulo some fixed positive integer. That is "20 modulo 9" is the value $r = \mathrm{mod}(20, 9)$ = remainder after dividing 20 by 9 where we have $20 = 2\cdot 9 + r$. We may write this $(20)_9 = 2$. $(20)_5 = 0$ because 5 divides 20 evenly with 0 remainder. Adding modulo a number n means

$$m_1 +_n m_2 = (m_1 + m_2)_n. \tag{1}$$

For example, $56 +_9 85 = (56 + 85)_9 = (141)_9 = 6$, because $141 = 15\cdot 9 + 6$. In order to reduce the level of arithmetic, notice that

$$m_1 +_n m_2 = ((m_1)_n + (m_2)_n)_n \tag{2}$$

So $56 +_9 85 = ((56)_9 + (85)_9)_9 = (2 + 4)_9 = (6)_9 = 6$.

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[1] Martin Gardner in his Introduction to [2] p.viii: "In recreational number theory he [Dudeney] was the first to apply "digital roots"—the term was probably coined by him—to numerous problems in which their application had riot been previously recognized as relevant."

A similar definition holds for multiplication:

$$m_1 \times_n m_2 = (m_1 \times m_2)_n \qquad (3)$$

For example, $56 \times_9 85 = (56 \times 85)_9 = (4760)_9 = 8$, because $56 \times 85 = (6{\cdot}9 + 2) \times (9{\cdot}9 + 4) = 9{\cdot}(6 + 9) + 2 \times 4)$. Again we have the arithmetic-reducing property:

$$m_1 \times_n m_2 = ((m_1)_n \times (m_2)_n)_n \qquad (4)$$

So $56 \times_9 85 = ((56)_9 \times (85)_9)_9 = (2 \times 4)_9 = 8$.

Casting Out 9s.

As the modulo 9 arithmetic suggested, there is a relationship with the digital root. The key is that $(10)_9 = 1$. Applying equation (4) yields $(10^n)_9 = ((10)_9{}^n)_9 = (1^n)_9 = 1$. Therefore, in any decimal expansion we have

$$(25936)_9 = (2{\cdot}10^4 + 5{\cdot}10^3 + 9{\cdot}10^2 + 3{\cdot}10 + 6)_9 = (2 + 5 + 9 + 3 + 6)_9 = (25)_9 = (2 + 5)_9 = 7$$

The one difference between the digital root and mod 9 arithmetic is that the digital root of 18 is 9, whereas $(18)_9 = (9)_9 = 0$. We can use the mod 9 arithmetic to simplify computing digital roots. So

$$(25936)_9 = (2 + 5 + 9 + 3 + 6)_9 = (2 + 5 + 0 + 3 + 6)_9 = (7 + 0)_9 = 7$$

This use of digital roots augmented with mod 9 arithmetic is referred to as “casting out 9s”.

Casting Out 3s

Mod 3 arithmetic inherits similar features from mod 9 arithmetic, that is, $(10^n)_3 = 1$ as well. Therefore we get

$$(25936)_3 = ((2)_3 + (5)_3 + (9)_3 + (3)_3 + (6)_3)_3 = (2 + 2 + 0 + 0 + 0)_3 = (4)_3 = 1$$

These ideas are very handy in solving digital puzzles.

Barrel of Beer Solution

Let W be the quantity of wine the man sold to the first man and let B be the amount of beer he kept for himself. Then, combined with twice the quantity of wine he sold to a second man, the total amount of beverage the man had was $N = B + 3W = 15 + 31 + 19 + 20 + 16 + 18$ gallons. Now $N - B = 3W$ implies $(N - B)_3 = (3W)_3 = ((3)_3\,(W)_3)_3 = (0{\cdot}(W)_3)_3 = 0$. So $(N)_3 = (B)_3$ Now

$$(B)_3 = (N)_3 = (15 + 31 + 19 + 20 + 16 + 18)_3 = (0 + 1 + 1 + 2 + 1 + 0)_3 = (2)_3 = 2$$

So the 20 gallon barrel holds the beer.

References

[1] Dudeney, Henry Ernest, *Amusements In Mathematics*, 1917

[2] Dudeney, Henry Ernest, *536 Puzzles & Curious Problems*, Edited By Martin Gardner, Charles Scribner's Sons, New York, 1967.