Angular Momentum

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This meditation differs from my other essays by not providing a solution or explanation for a conundrum that I came across. Rather it presents the issues I have with the idea of angular momentum (a.k.a. moment of momentum[1]) without providing a resolution. I have always had a tenuous relationship with the concept, but recently my concerns resurfaced when I did my studies on Kepler, and in particular his "equal areas law" and Newton's elegant geometric proof. I love the fact that a simple geometric argument, seemingly totally divorced from the physical situation, can provide an explanation for why the line from the Sun to a planet sweeps out equal areas in equal time as the planet orbits the Sun, solely under the influence of the gravitational force between them.

However, modern physics books invariably cite the conservation of angular momentum as the "explanation." This recalls to my mind the admonition that Feynman's father made to his son that just putting a label or name to something does not constitute an explanation. To aggravate the situation, the modern texts go on to explain angular momentum with one of the more abstruse and unintuitive constructs from the otherwise clarifying use of vectors, namely, the vector cross product. This "product" of vectors does indeed produce another vector, but it is *perpendicular* to the two vectors being multiplied! Where did that come from? Oh, it arose from abstract 19th century number system called *quaternions*, which was a 4-dimensional extension of the 2-dimensional complex numbers—already a stretch of the imagination from familiarity with real numbers. So we have layer upon layer of abstract concepts, all used to "explain" the planetary motion by means other than the equal areas law. (Strangely, there is a link between the vector cross product and areas, namely, the "length" or magnitude of the resulting perpendicular vector is the area of the parallelogram spanned by the two vectors being multiplied. So area does enter the picture implicitly.)

But there is more to the angular momentum explanation for planetary motion than vectors and cross products: there is a concept of *rotational* momentum in contrast to *linear* momentum, with the further claim that the angular momentum of an entity (rigid object, system of objects, liquid, gas, etc.) is conserved. These are deep ideas. It took millennia for thinkers to arrive at the idea of the conservation of linear momentum (law of inertia). It became Newton's First Law. So how did the idea of angular momentum and its conservation arise?

I wish to avoid a lengthy mathematical discussion of the sort found in current physics books and concentrate on some fundamental ideas, in particular those that were historically associated with early attempts to understand the concept.

History

One historian of science has been particularly helpful, namely, Clifford Truesdell in his 1964 essay "Whence the Law of Momentum of Momentum?" ([2]) and his earlier 1960 essay "A Program toward Rediscovering the Rational Mechanics of the Age of Reason" ([1]). In "Whence the Law…" he states ([2] p.594):

> "… a historian of science … will not expect any development of mechanics as a whole along "Newtonian" lines by Newton or anyone else before the middle of the nineteenth century, nor will

[1] JOS: From Feynman ([4] p.18-5): The origin of this term is obscure, but it may be related to the fact that "moment" is derived from the Latin *movimentum*, and that the capability of a force to move an object (using the force on a lever or crowbar) increases with the length of the lever arm. In mathematics "moment" means weighted by how far away it is from an axis.

he expect the *general* principle of moment of momentum to be stated before its first great application, namely, the theory of general motion of a rigid body, was discovered."

But Truesdell does acknowledge Newton's awareness of angular momentum in a footnote ([2] p.594n):

> This fact does not imply that Newton had never thought about these matters, as is shown by the famous comment following Law I: "A top, whose parts by their cohesion constantly draw each other had, from rectilinear motions, does not stop spinning except in so far as it is slowed by the air." It would he interesting to see if among Newton's papers there is anything mathematical concerning the motion of rigid bodies.

But Truesdell makes clear that there is no evidence in the extant writings of Newton that he felt that rotational motion was covered by his laws, contrary to the assertions of subsequent philosophers of physics, especially in the late 19th century, where they claimed (falsely) the general laws of angular momentum could be derived from "Newton's equations," as Truesdell would put in quotes.

So Truesdell asserts that the origin of angular momentum ideas arose from the study of *rigid* bodies, some even pre-dating Newton with the work of Huygens on pendulums and James Bernoulli on moments and lever arms. It did not derive from Newton's laws of gravitational attraction and planetary motion. In describing the contributions to angular momentum by Euler in the 18th century in "Rediscovering the Rational Mechanics," Truesdell states ([1] p.32n):

> EULER sought later an approach to mechanics as a whole that would yield directly and easily the equations of motion of a rigid body as a special case. This he achieved in a memoir published in 1776, where he laid down the following laws as applicable to *every part of every body,* whether punctual or space-filling, whether rigid or deformable:
>
> Law 1. The total force acting upon the body equals the rate of change of the total momentum.
>
> Law 2. The total torque acting upon the body equals the rate of change of the total moment of momentum, where both the torque and the moment are taken with respect to the same fixed point.
>
> … The law of moment of momentum is subtle, often misunderstood even today:
>
> In presentations of mechanics for physicists it is usually derived as a consequence of "NEWTON'S laws" for elements of mass which are supposed to attract each other with mutual forces which are central and pairwise equilibrated. Although the formal steps of this derivation are correct, the result is too special for continuum mechanics as well as methodologically wrong for rigid mechanics:
>
> 1. Any forces between the particles of a rigid body never manifest themselves, by definition, in any motion. The condition of rigidity applied to EULER'S second law suffices to determine the motion. To hypothecate mutual forces is to luxuriate in superfluous causes, which are to be excised by OCKHAM'S razor.
> 2. To introduce mutual forces in a rigid body drags in action at a distance in a case when it is unnecessary to do so.
> 3. In continuum mechanics the total force acting upon a finite body arises principally from the stress tensor, which represents the contiguous action of material on material. There is no physical reason to assume that forces arise only from action at a distance, and no purpose served by doing so. If rigid-body mechanics is viewed as a special case of continuum mechanics, the stress within the rigid body is indeterminate and need never be mentioned, but a uniform process based upon EULER'S second law remains possible.

No work of EULER, nor of any other savant of the eighteenth century, approaches rigid bodies by hypothecating forces acting at a distance as in modern books. EULER'S method of discovery tacitly assumes there to be no internal forces at all; the right answer comes out, but we are justified in doubting that the argument be general enough. His final treatment does not make any presumption or restriction regarding the presence or nature of internal forces.

So at this point, according to Truesdell, I feel vindicated that the idea of angular momentum was not part of the original Newtonian description of planetary motion, that the idea apparently developed in parallel originally based on the motion of rigid bodies involving moments (radial distance) by James Bernoulli and contemporaries, and that it came to fruition about a century after Newton in the work of Euler and others. In particular, as Truedell's quote regarding Euler suggests, the *physical* association of rigid-body angular momentum with planetary motion is specious. So why is angular momentum associated with planetary motion after all?

I claim it is because they have the same *mathematical* expression, and in particular, they both involve the equal areas property reflected in the mathematics. In fact, Lagrange in the late 1700s still refers to angular momentum notions in terms of "equal areas" ([2] p.595).[2]

Modern Approach

Perhaps the best support for my argument comes from Richard Feynman's *Lectures on Physics* ([4]) that were part of my early education in physics. Chapters 18-20 cover rotational motion and angular momentum.

Torque. Feynman begins with the rotation of rigid bodies and, in particular, the planar rotation of a mass particle a fixed distance about an axis perpendicular to the plane. Following Newtonian principles, he arrives at a mathematical expression of torque. As he puts it ([4], p.18-4c):

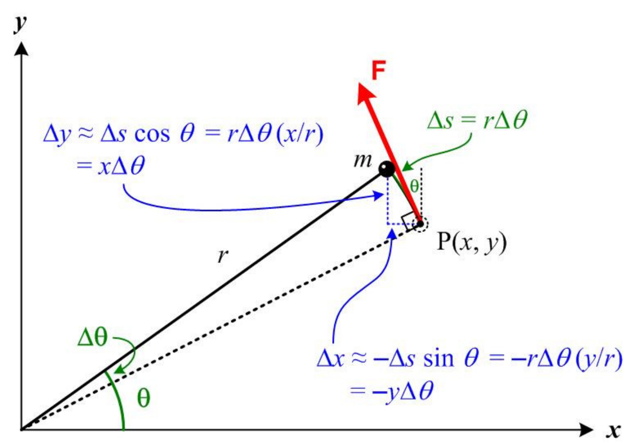


Figure 1 Motivation for Torque

> Let us inquire whether we can invent something which we shall call the *torque* (L. *torquere*, to twist) which bears the same relationship to rotation as force does to linear movement. A force is the thing that is needed to make linear motion, and the thing that makes something rotate is a "rotary force" or a "twisting force," i.e., a torque. Qualitatively, a torque is a "twist"; what is a torque quantitatively? We shall get to the theory of torques quantitatively by studying the *work* done in turning an object, for one very nice way of defining a force is to say how much work it does when it acts through a given displacement.

Assume some mass m is located at a point P(x, y) on a rigid, weightless rod of length r pivoting around the origin as shown in Figure 1, and a force F is applied perpendicular to the rod at P to move the mass an incremental distance Δs along an arc. (Since the rod is rigid, the only direction for the force to cause motion is perpendicular to the rod.) Then the incremental work done is $\Delta W = F\ \Delta s = F\ r\Delta\theta$. The quantity that corresponds to a rotational force acting through a rotation $\Delta\theta$ is rF. This quantity is called the *torque* τ and is measured as the perpendicular force times the distance from the

[2] JOS: (Update 8/1/2018) For a more formal history of the evolution of the idea of "angular momentum" from the Kepler-Newton notion of equal areas to the coining of the term in 1856 by Robert Baldwin Hayward, see Arianna Borrelli's article "Angular Momentum Between Physics And Mathematics" [3]

pivot (for a perpendicular force this distance to the pivot is called the *lever arm*, connoting a physical connection like a crow bar). Using rectangular coordinates and decomposing the force F into its horizontal and vertical components, F_x and F_y, respectively, Figure 1 shows the expression for work can be written as the sum of the contributions from these components:

$$\Delta W = F\,\Delta s = F_x\,\Delta x + F_y\,\Delta y = xF_y\,\Delta\theta - yF_x\,\Delta\theta$$

Thus, as Feynman explains, the torque τ becomes

$$\tau = rF = xF_y - yF_x \tag{1}$$

I like to think of the situation in the following way. That the rotation angle θ plays a central role can be more clearly understood when we consider additional masses located on rigid spokes of various lengths emanating from the central pivot or hub, as in Figure 2. If as before, we apply a force to the mass m_1 and move it along the arc Δs_1, then the rod will rotate an angle $\Delta\theta$. But that means the force we apply also has to move the larger mass m_2 over a longer arc Δs_2 (and faster), which will take more force. So the total rotational force ("total torque" τ) we have to apply to rotate the entire rod and masses through an angle $\Delta\theta$ is the sum of the separate torques:

$$\tau = \tau_1 + \tau_2 = r_1F_1 + r_2F_2$$

Because of the common rotation angle, the amount of forces required depends on the distances from the pivot point, and thus the necessity for "moments" or including the radial distance in the idea of a rotational force.

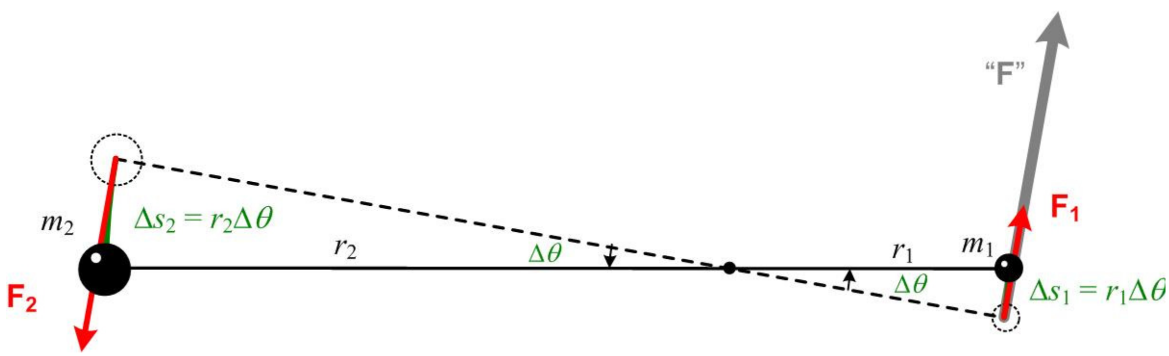


Figure 2 Total Torque

As an interesting footnote, Kepler innovatively believed that the sun was somehow responsible for the motion of the planets. He imagined there was some type of radial attractive force, like magnetism, between the planet and the sun, but he also believed that there had to be some type of transverse force that the sun exerted on the planets to get them to move in their orbits, like a rigid rod sweeping the planets on their way (this was before the acceptance of the law of inertia). So he may have intuitively had some notion of moments in doing his calculations.

Angular Momentum. Now back to Feynman's exposition. He proceeds to derive a mathematical expression which he defines as angular momentum ([4] pp.18-5 – 18-6) (my highlighting):

> Although we have so far considered only the special case of a rigid body, the properties of torques and their mathematical relationships are interesting also even when an object is not rigid. In fact, we can prove a very remarkable theorem: just as external force is the rate of change of a quantity p, which we call the total momentum of a collection of particles, so the external torque is the rate of change of a quantity L which we call the *angular momentum* of the group of particles.

To prove this, we shall suppose that there is a system of particles on which there are some forces acting and find out what happens to the system as a result of the torques due to these forces. First, of course, we should consider just *one* particle. In Fig. 18–3 is one particle of mass m, and an axis O; the particle is not necessarily rotating in a circle about O, it may be moving in an ellipse, like a planet going around the sun, or in some other curve. It is moving somehow, and there are forces on it, and it accelerates according to the usual formula that the x-component of force is the mass times the x-component of acceleration, etc. But let us see what the *torque* does. The torque equals $xF_y - yF_x$, and the force in the x- or y-direction is the mass times the acceleration in the x- or y-direction:

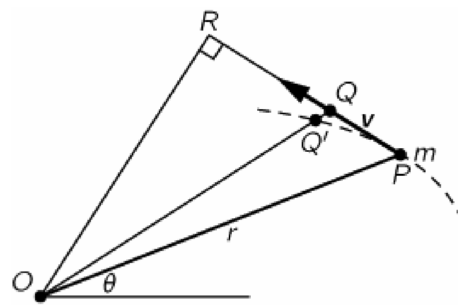


Fig. 18–3. A particle moves about an axis O.

$$\begin{aligned}\tau &= xF_y - yF_x \\ &= xm(d^2y/dt^2) - ym(d^2x/dt^2)\end{aligned} \tag{18.14}$$

Now, although this does not appear to be the derivative of any simple quantity, it is in fact the derivative of the quantity $xm(dy/dt) - ym(dx/dt)$:

$$\begin{aligned}\frac{d}{dt}\left[xm\left(\frac{dy}{dt}\right) - ym\left(\frac{dx}{dt}\right)\right] &= xm\left(\frac{d^2y}{dt^2}\right) + \left(\frac{dx}{dt}\right)m\left(\frac{dy}{dt}\right) \\ &\quad - ym\left(\frac{d^2x}{dt^2}\right) - \left(\frac{dy}{dt}\right)m\left(\frac{dx}{dt}\right) = xm\left(\frac{d^2y}{dt^2}\right) - ym\left(\frac{d^2x}{dt^2}\right)\end{aligned} \tag{18.15}$$

[p.18-6] So it is true that the torque is the rate of change of something with time! So we pay attention to the "something," we give it a name: we call it L, the angular momentum:

$$\begin{aligned}L &= xm(dy/dt) - ym(dx/dt) \\ &= xp_y - yp_x.\end{aligned} \tag{18.16}$$

Therefore, the torque τ is dL/dt or the time rate of change of angular momentum. An important instance of $\tau = dL/dt$ is the ***law of conservation of angular momentum****:* if no external torques act upon a system of particles ($\tau = 0$), the angular momentum remains constant ($dL/dt = 0 \Rightarrow L = \text{constant}$).

So a mathematical expression for something called torque, based on the rotation of a rigid body around a fixed axis, is extended to the planar motion of an arbitrary body with respect to a fixed point, from which is defined a mathematical expression called angular momentum, and a law called conservation of angular momentum. It seems that Feynman *motivates* his concepts via Newtonian principles, but then *defines* them as independent mathematical expressions, more or less following the method of Euler, where he makes explicit by his *two* Law above (p.2) that the laws of rotational motion are inherently independent of those for linear motion. These mathematical expressions are then shown to arise in situations not associated with rigid motion, as Feynman indicates. We shall see how this might be in the case of planetary motion.

Planar Motion

First, to streamline the presentation, we introduce a vector description of motion in a plane, which includes the strange vector cross product.

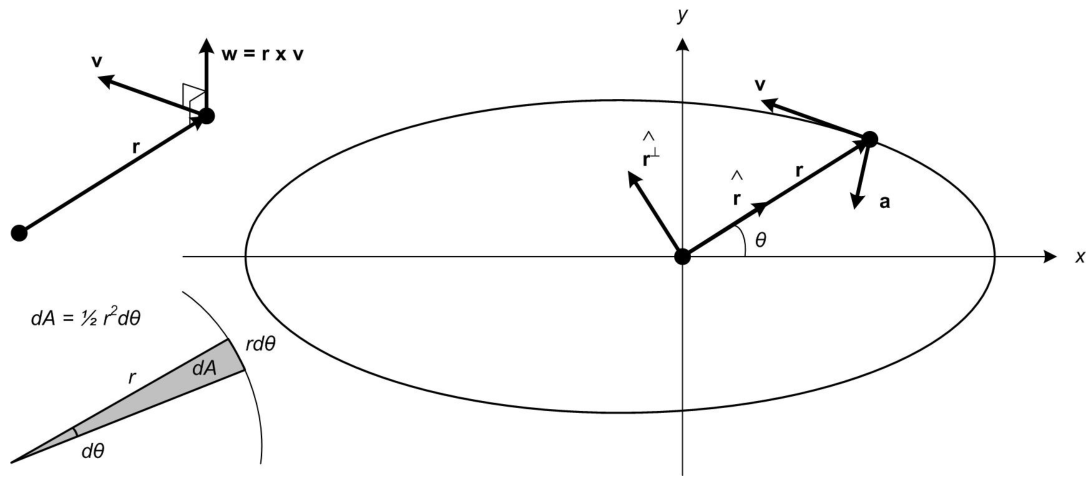


Figure 3 Vector and Area Relationships

Using the diagrams in Figure 3, we have, for the area A swept out by the position vector $\mathbf{r}$ to a moving particle, the following time rate of change derivatives

$$\dot{A} = \tfrac{1}{2} r^2 \dot{\theta} \quad \text{implies} \quad \ddot{A} = r\dot{r}\dot{\theta} + \tfrac{1}{2} r^2 \ddot{\theta} \tag{2}$$

where the dots over the variables represent time derivatives (double dots, the second derivative).

Now we obtain the general polar coordinate vector equations for arbitrary motion in a plane about a fixed point. Let $\hat{\mathbf{r}}$ represent the unit vector along the position vector $\mathbf{r}$ and $\hat{\mathbf{r}}^{\perp}$ the counterclockwise unit vector perpendicular to $\mathbf{r}$ in the plane (see Figure 3), namely, (in Cartesian coordinates with basis vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$)

$$\hat{\mathbf{r}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta \quad \text{and} \quad \hat{\mathbf{r}}^{\perp} = -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta$$

so that

$$\dot{\hat{\mathbf{r}}} = \hat{\mathbf{r}}^{\perp}\dot{\theta} \quad \text{and} \quad \dot{\hat{\mathbf{r}}}^{\perp} = -\hat{\mathbf{r}}\dot{\theta}$$

Now the velocity vector $\mathbf{v}$ is given by

$$\mathbf{v} = \dot{\mathbf{r}} = \hat{\mathbf{r}}\dot{r} + \hat{\mathbf{r}}^{\perp} r\dot{\theta}$$

and the acceleration vector $\mathbf{a}$ by

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{r}}^{\perp}$$

so that in terms of the expressions for the area swept out by the position vector $\mathbf{r}$ given in equations (2), we have

$$\boxed{\mathbf{a} = (\ddot{r} - \frac{4\dot{A}^2}{r^3})\hat{\mathbf{r}} + \frac{2\ddot{A}}{r}\hat{\mathbf{r}}^{\perp}} \tag{3}$$

So right away in the general mathematical expressions for motion in a plane relative to a fixed point, we see expressions involving the time rate of change of the area swept out by the moving position vector.

Angular Momentum – Vector Version

We now convert the numerical expressions Feynman gave above for torque and angular momentum into vector expressions, as his book also did in Chapter 20. The explanation for the cross product is given below in an Appendix (p.11). From the Appendix equation (A.3),

$$\boldsymbol{\tau} = \tau\mathbf{k} = (xF_y - yF_x)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ F_x & F_y & 0 \end{vmatrix} = \mathbf{r} \times \mathbf{F}.$$

Notice that in general the force **F** does not have to be perpendicular to the position vector **r**, but the trigonometric form of the cross product ($|\mathbf{r}||\mathbf{F}| \sin\theta\,\mathbf{k}$) shows we are effectively multiplying the length of the position vector r by the component of **F** perpendicular to **r**, namely $F \sin\theta$. And if **F** is actually perpendicular to **r** ($\sin\theta = 1$), the magnitude of the torque τ is rF, just as before.

Furthermore,

$$(xmv_y - ymv_x)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & 0 \\ mv_x & mv_y & 0 \end{vmatrix} = \mathbf{r} \times m\mathbf{v}.$$

where the components of the velocity vector $\mathbf{v} = \dot{\mathbf{r}}$ are given by $v_x = dx/dt$ and $v_y = dy/dt$. The derivation of Feynman equation (18.15) with vectors and the vector product becomes simply:

$$\frac{d}{dt}(\mathbf{r}\times m\mathbf{v}) = \dot{\mathbf{r}}\times m\mathbf{v} + \mathbf{r}\times m\dot{\mathbf{v}} = m\overset{\mathbf{0}}{\overbrace{(\mathbf{v}\times\mathbf{v})}} + \mathbf{r}\times m\mathbf{a} = \mathbf{r}\times m\mathbf{a} \tag{4}$$

So as defined above by Feynman in equation (18.16), angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$ (or $\mathbf{r} \times \mathbf{P}$ where **P** represents linear momentum $m\mathbf{v}$) and equation (4) imply $\boldsymbol{\tau} = d\mathbf{L}/dt = d(\mathbf{r} \times m\mathbf{v})/dt$ (torque = rate of change of angular momentum), which is strongly analogous to $\mathbf{F} = d\mathbf{P}/dt = d(m\mathbf{v})/dt$ (force = rate of change of linear momentum). Similarly, $\boldsymbol{\tau} = d\mathbf{L}/dt = 0$ (conservation of angular momentum) implies the angular momentum **L** remains constant, and $\mathbf{F} = d\mathbf{P}/dt = 0$ (conservation of linear momentum) implies the linear momentum **P** remains constant (law of inertia).

Now we finally re-introduce area into the picture via equations (3) and (4):

$$\boldsymbol{\tau} = d\mathbf{L}/dt = \mathbf{r} \times m\mathbf{a} = m\left(\ddot{r} - \frac{4\dot{A}^2}{r^3}\right)\overset{\mathbf{0}}{\overbrace{\mathbf{r} \times \hat{\mathbf{r}}}} + m\frac{2\ddot{A}}{r}\, r\hat{\mathbf{r}} \times \hat{\mathbf{r}}^{\perp} = m\,(2\ddot{A})\,\mathbf{k}$$

Hence, considering only the magnitudes of the vectors, we have

$$\tau = m2\ddot{A} \tag{5}$$

Here we have explicit evidence tying torque and angular momentum to areas. Therefore, $\tau = 0$ (Conservation of Angular Momentum) if and only if $\ddot{A} = 0$ if and only if dA/dt = constant (the position vector **r** sweeps out equal areas in equal times, i.e., Equal Areas Law). So

Conservation of Angular Momentum is equivalent to the Equal Areas Law.

There is admittedly a lot of hand-waving going on. The explicit use of a position vector and force vector in the expression of torque is somewhat ambiguous in the case of extended bodies and deformable media. But in the two cases of interest, planetary and rigid body motion, the expression seems to make sense.

Planetary Motion (Central Force). In the case of planetary motion, we have that the force of gravity $\mathbf{F}$ is a central force, that is, it is directed along the position vector $\mathbf{r}$. Therefore, $\mathbf{F} \parallel \mathbf{r}$ and that means $\mathbf{r} \times \mathbf{F} = \mathbf{0}$, which means the torque $\tau = 0$, and thus the law of equal areas holds.

Rigid Body. In the case of rigid body rotation, the length of any position vector r is constant, so $\dot{r} = 0$. From equation (4) this means $\ddot{A} = \frac{1}{2} r^2 \ddot{\theta}$ and from (5) that $\tau = mr^2 \ddot{\theta}$. So the torque is 0 if and only if the rigid body is rotating at a constant rate $\dot{\theta} = \omega$. But that means $\dot{A} = \frac{1}{2} r^2 \dot{\theta}$ is constant and we again have the law of equal areas.

So here we have two cases satisfying the equal areas law, planetary motion mysteriously by Newton's geometric proof and rigid body motion obviously by constant circular rotation, that also happen to be tied together via the no less mysterious mathematical construct of vanishing torque or equivalently conservation of angular momentum.

Now I recognize, as Newton did, that there seems to be a *physical* principle of angular momentum and its conservation, at least with rigid bodies, such as wheels. This MIT demonstration on Youtube[3] is most convincing:

Figure 4 MIT Physics Demonstration of Conservation of Angular Momentum

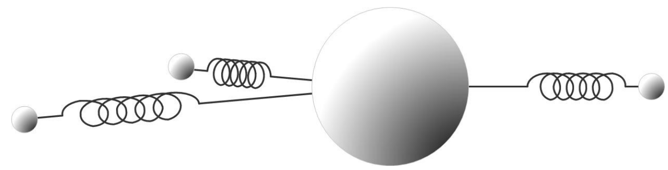
But the *physical* connection with planetary motion is not at all clear. That is, contrary to Kepler, there is no crow bar or physical connection between the sun and the transversally moving planet, so why should we use the same explanation as in a rigid body rotation? This is why I don't understand the *physical* "explanation" of planetary motion as due to conservation of angular momentum. Certainly, as we saw above, Truesdell does not believe planetary motion (based on Netwon's Laws) is a physical explanation for rigid body rotation. And I don't believe the reverse, that physical concepts derived from the rotation of rigid bodies offer a physical explanation for planetary motion. Apparently, the only connection between these two situations is a set of mathematical expressions.

[3] 150615 MIT Physics Demos Conservation of Angular Momentum.mp4 (https://www.youtube.com/watch?v=_XgYTP0kB7A, retrieved 4/13/17)

Common Mathematical Equations ⇒ Common Physical Phenomena?

So we are confronted with the idea in physics that if two seemingly separate phenomena have common mathematical expressions, then they must be physically linked as well (even if no physical link is evident). This is a profound idea and elevates Wigner's "Unreasonable Effectiveness of Mathematics in the Natural Sciences" ([7]) to a whole new level. However, it does lead to some strange situations.

Simple Harmonic Motion. A great example is the intrusion of simple harmonic motion into the mix of planetary motion. When viewed sideways, as we do from earth, the motion of the moons about Jupiter appear to oscillate in simple harmonic motion, as if connected by springs (at least that was Galileo's thought).



Why this is so can be seen closer to earth by imagining the motion of a satellite skimming around just above the surface of the earth in perfectly circular orbit, as shown in Figure 5. Viewing the situation edge-on along the x-axis essentially reveals only the vertical projection of the satellite's motion along the y-axis. The effect of the force **F** of gravity is then only its y-component F_y. But assuming a constant rotation rate ω, yields

$$F_y = F\cos\theta \propto -r\cos\theta\,\omega^2$$

or

$$F_y = -k_1\,y,$$

which is the equation for simple harmonic motion.

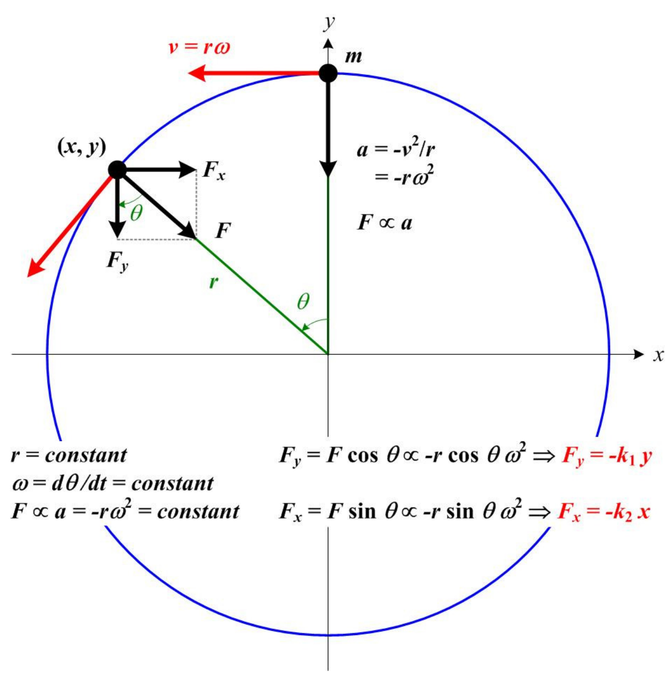


Figure 5 Two Simple Harmonic Motion Cases

What is even more astonishing, is the famous behavior of dropping the same satellite through an imaginary hole through the earth to the other side (Figure 5). (We suppose for this fantasy that the earth is of uniform mass density throughout.) The effect of gravity on the falling mass from parts of the earth more distant from the center than the satellite magically cancel out so that only the mass of the earth within the sphere of radius equal to the satellite's instantaneous distance from the center counts. The resulting effect is that the force diminishes as the satellite approaches the center in exactly the same simple harmonic form as the vertical motion of the orbiting satellite (so that amazingly both satellites reach the antipodal point at the same time!). A marvelous demonstration of all this is given by a Minute Physics cartoon on Youtube:[4]

[4] 160523 How Long To Fall Through Earth.mp4 (https://www.youtube.com/watch?v=urQCmMiHKQk, retrieved 5/11/17)

Figure 6 Minute Physics: How Long To Fall Through Earth

So does that mean planetary motion is really simple harmonic motion? or that simple harmonic motion is really planetary motion? because they have the same mathematical expressions? Another Minute Physics Youtube video gives an insightful presentation of the idea of multiple mathematical models for the same phenomena:[5]

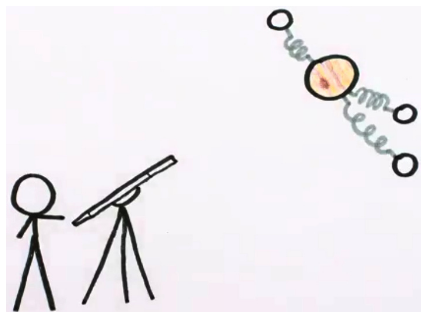


Figure 7 Minute Physics: How Perspective Shapes Reality

And then there is Max Tegmark's view that physical reality *is* mathematics, which would mean all *physical* explanations were merely labeling inherent mathematical formulations:[6]

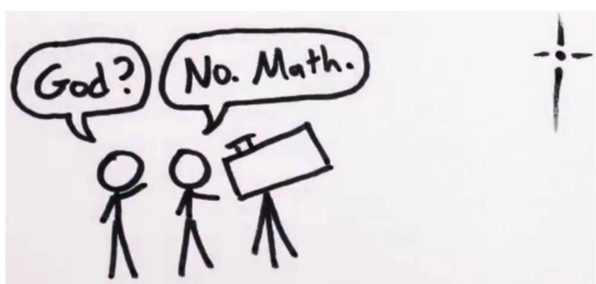


Figure 8 Minute Physics: Is the Universe Entirely Mathematical - Max Tegmark

Feynman Again

Update (5/30/2018) By one of those strange coincidences, I happened to be reviewing the Youtube recording of the Feynman 1965 Messenger Lectures I attended at Cornell that were transcribed in the book *The Character of Physical Law* ([10]). The entire second lecture (second chapter), called "The Relation of Mathematics to Physics" addresses the very questions I was raising in this essay. As intimated in my excerpts from Feynman's Lecture Notes ([4]), Feynman

[5] 161117 How Perspective (Models) Shapes Reality.mp4 (https://www.youtube.com/watch?v=lHaX9asEXIo, retrieved 4/23/17)

[6] 140526 Is the Universe Entirely Mathematical - Max Tegmark.mp4 (https://www.youtube.com/watch?v=HGG4HmlotJE, retrieved 4/24/17) See also references [8] and [9].

emphasized in the lectures the centrality of mathematics in describing physics, as well as its ability to reveal relationships that were not evident before. In particular, Feynman argues that the concept of angular momentum was basically a name assigned to common mathematical expressions for equal areas that showed up in different physical situations. He also mentioned how the physical example of rigid body rotation was different from Newton's gravitational interactions among multiple masses, but that they led to the same mathematical expression, which was elevated to a general physical principle called "angular momentum." "(This is the same guessing game as taking the angular momentum and extending it from one case where you have proved it, to the rest of the phenomena of the universe.)" Feynman admitted the mystery behind these connections, but claimed they were real, nevertheless, and were at the heart of the pursuit to understand nature. One just had to accept the intimate presence of mathematics in virtually all physical explanations of nature. Earlier Feynman had discussed how every effort to provide a *physical* cause-and-effect explanation for gravity failed. All that we had left were the mathematical equations. So I guess I was on the right track with my observations about angular momentum and its larger implications.

Conclusion

In my waning years as I probe more closely scientific ideas originally presented in my youth, I find I understand things less and less. Both in classical physics and the modern physics of quantum mechanics, the physical causal relationships seem to fade and the mathematical formulations seem to intensify. That has the strange effect of making "physical reality" become more intangible and subject to human imagination in the form of mathematics. Of course, reality is actually quite tangible and "real," so the result of these ruminations is a sort of cognitive dissonance that can be most disorienting—or possibly exhilarating with its delightful mystery and marvel.

Appendix: Vector Product

We provide a quick review of the vector cross product (and the dot product, as well). Kline ([5]) has a nice summery of the development of vector analysis by Gibbs and Heaviside in the 1880s from Hamilton's earlier invention of quaternions in the mid 1800s. As explained by Hamilton's student P. G. Tait in his 1867 treatise on quaternions ([6]), the ideas of the scalar product and cross product of vectors in both their coordinate form and trigonometric form were already derived from the operations on quaternions and their geometric association with rotations in 3-dimensional space. And the representation of the vector cross product via a determinant was also given in Tait's 1867 book.

To briefly review: The *scalar* or *dot product* of two 3-dimensional vectors $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ is derived from the definition of the scalar product for the basis vectors:

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \quad \text{and} \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0,$$

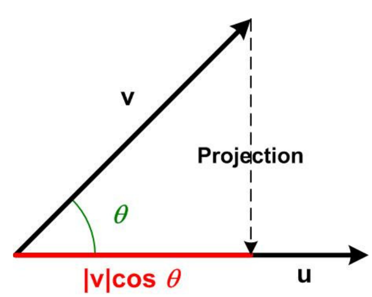
and the commutative and distributive laws (all of which originally derived from quaternion multiplication). Therefore,

$$\mathbf{u}\cdot\mathbf{v} = ad + be + cf$$

The length of a vector $\mathbf{u}$, designated $|\mathbf{u}|$, becomes $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$.

Geometric Interpretation. Furthermore, if θ is the angle between the two vectors $\mathbf{u}$ and $\mathbf{v}$, then the scalar product can also be written

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta.$$



This can be interpreted as multiplying the length of $\mathbf{u}$ by the orthogonal projection of the length of $\mathbf{v}$ onto $\mathbf{u}$ (or multiplying the length of **v** by the orthogonal projection of **u** onto **v**). Notice that if $\mathbf{u}\cdot\mathbf{v} = 0$, then **u** and **v** are orthogonal (the angle between them is 90°).

The *vector* or *cross product* of two vectors also derives from its definition for the basis vectors:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \text{and} \quad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

the distributive laws, and an anti-commutative law: $\mathbf{u} \times \mathbf{v} = -\,\mathbf{v} \times \mathbf{u}$. Unlike the rationals, the reals, and the complexes, which all have a commutative rule, the quaternions do not, even though they share all other properties, such as a zero, an identity, and additive and multiplicative inverses. And the derived vector product loses one more property, a really basic one, namely the associative rule! As can easily be seen,

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \text{but} \quad \mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$$

So this cross product requires a lot of care in its use. The component version of the cross product using $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + f\mathbf{k}$ is:

$$\mathbf{u} \times \mathbf{v} = (bf - ce)\mathbf{i} + (cd - af)\mathbf{j} + (ae - bd)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix} \tag{A.1}$$

showing a nice representation using determinants (which is the only way to keep things straight). Combining the dot product of a vector $\mathbf{w} = r\mathbf{i} + s\mathbf{j} + t\mathbf{k}$ with the cross product $\mathbf{u} \times \mathbf{v}$ yields

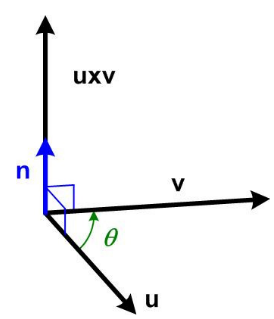
$$\mathbf{w}\cdot(\mathbf{u} \times \mathbf{v}) = r(bf - ce) + s(cd - af) + t(ae - bd) = \begin{vmatrix} r & s & t \\ a & b & c \\ d & e & f \end{vmatrix} \tag{A.2}$$

From this determinant representation we can easily see that $\mathbf{w}\cdot(\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \times \mathbf{u})\cdot\mathbf{v}$, since both expressions yield the same determinant. And from this we can see the cross product $\mathbf{u} \times \mathbf{v}$ is perpendicular to both **u** and **v**, since $\mathbf{u}\cdot(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{u})\cdot\mathbf{v} = 0$ and $\mathbf{v}\cdot(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v} \times \mathbf{v}) = 0$. This cross product is indeed a strange animal, but basically it is just a simple way to write some complicated expressions.

Geometric Interpretation. Again, if θ is the angle between the two vectors **u** and **v** (this time explicitly measured from **u** to **v**), then the cross product can also be written

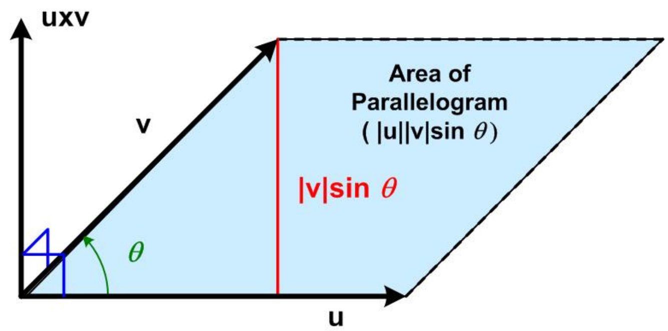
$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta\, \mathbf{n}$$

where the vector **n** is a unit vector normal (perpendicular) to the plane through **u** and **v**, in accordance with the "right-hand rule." That is, if the thumb on the right hand represents **u** and the first finger represents **v**, then the middle finger folded perpendicular to the palm represents **n** and thus the cross product $\mathbf{u} \times \mathbf{v}$.



And similar to the dot product, if the cross product vanishes, we know something about the orientation of the vectors **u** and **v**, namely, that they are parallel, with possibly the opposite orientation (the angle between them is either 0° or 180°).

There is an additional geometric interpretation hidden in the trigonometric expression for the cross product. It provides the area of the parallelogram spanned by the vectors **u** and **v**. So besides being able to encompass some complicated numerical expressions, the cross product includes the idea of a perpendicular vector to two other vectors, as well as a notion of area. So the innocuous looking product has a very powerful representation capability.



Finally, in the special case where we can consider the plane determined by the vectors **u** and **v** to be the *xy*-plane, then $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + 0\mathbf{k}$ and $\mathbf{v} = d\mathbf{i} + e\mathbf{j} + 0\mathbf{k}$ and the cross product takes the simple form:

$$\mathbf{u} \times \mathbf{v} = (ae - bd)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 0 \\ d & e & 0 \end{vmatrix}. \qquad \text{(A.3)}$$

And so the value of $|\mathbf{u} \times \mathbf{v}|$ is just $|(ae - bd)\mathbf{k}| = (ae - bd)|\mathbf{k}| = ae - bd$.

References

[1] Truesdell, C., "A Program toward Rediscovering the Rational Mechanics of the Age of Reason," *Arch. Hist. exact Sci.*, Vol. 1, **1960**. pp.3-36 (https://eclass.upatras.gr/modules/document/file.php/CIV1558/%CE%99%CF%83%CF%84%CE%BF%CF%81%CE%B9%CE%BA%CE%AC/Truesdell.pdf)

[2] Truesdell, Clifford, "Whence the Law of Momentum of Momentum?" in: Alexandre Koyre, ed., *L'Aventure de la Science*, *Histoire de la Pensee*, vol. 1 (Hermann, Paris), **1964**, pp.149-158 [588-612?] (www.math.cornell.edu/~rand/randdocs/classics/truesdell%20whence.pdf, retrieved 3/11/2017) Slightly revised version as Chapter V in Clifford Truesdell, *Essays in the History of Mechanics*, Springer, New York, 1968.

[3] Borrelli, Arianna, "Angular Momentum Between Physics And Mathematics," in Karl-Heinz Schlote and Martina Schneider (eds.), *Mathematics Meets Physics* (Frankfurt a. M.: Verlag Harri Deutsch, **2011**) 395-440. (www.weatherglass.eu/PDFs/Angular_momentum.pdf, retrieved 3/10/2017)

[4] Feynman, Richard, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics, Volume I, Mainly Mechanics, Radiation, and Heat*, Addison-Wesley Publishing Co, Reading, Massachusetts, **1963**. (http://www.feynmanlectures.caltech.edu/) Chapters 18-20.

[5] Kline, Morris, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, **1972** (https://archive.org/details/MathematicalThoughtFromAncientToModernTimes, retrieved 5/26/2017), Chapter 32

[6] Tait, Peter Guthrie, *Elementary Treatise on Quaternions*, Clarendon Press, Oxford, **1867** (https://archive.org/download/anelementarytre03taitgoog/anelementarytre03taitgoog.pdf, retrieved 3/16/2017)

[7] Wigner, Eugene, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," in *Communications in Pure and Applied Mathematics*, vol. 13, No. I (February **1960**). New York: John Wiley & Sons, Inc. Copyright © 1960 by John Wiley & Sons, Inc. (http://www.dartmouth.edu/~matc/MathDrama/reading/Wigner.html, retrieved 9/29/2002)

[8] Tegmark, Max, *Our Mathematical Universe: My Quest for the Ultimate Nature of Reality*, Vintage Books, **2014**.

[9] Tegmark, Max, *Our Mathematical Universe with Max Tegmark* (45 min), (http://richannel.org/our-mathematical-universe1, 4/11/17) and *Our Mathematical Universe Q and A with Max Tegmark* (41 min), (http://richannel.org/qampa--our-mathematical-universe, retrieved 4/11/17), Ri Channel (Royal Institution), 8 October **2014**.

[10] Feynman, Richard, "The Relation of Mathematics to Physics," *The Character of Physical Law,* Chapter 2, MIT Press, **1965**. First MIT Press Paperback Edition, March 1967, 12th printing, 1985. pp.35-58 (people.virginia.edu/~ecd3m/1110/Fall2014/The_Character_of_Physical_Law.pdf) Youtube Lecture 2: https://www.youtube.com/watch?v=kd0xTfdt6qw