Vitruvian Man Problem

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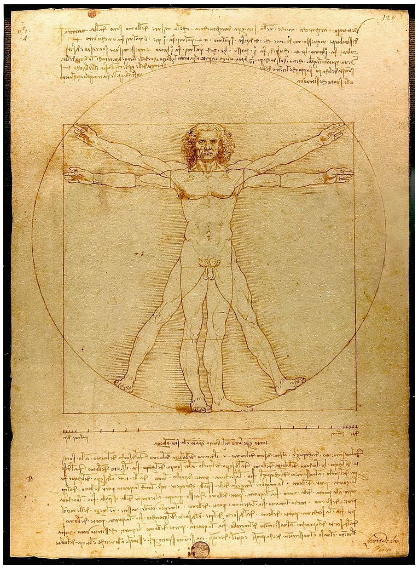

Figure 1 Leonardo Da Vinci's Vitruvian Man

I started reading the *New York Times* review [1] of Toby Lester's latest book, *Da Vinci's Ghost* [2],[1] but stopped when I saw the picture of Da Vinci's famous drawing of the Vitruvian Man (Figure 1). A curious mathematical question came to mind about the drawing. I am sure this question has been asked before (and answered), but I thought I would consider it on my own.

Figure 2 shows the Vitruvian Man overlaid with the circle and the points of contact on the circle made by the spread-eagled and at-rest figure of the man. Only the raised arms and both the at-rest and raised feet touch the circle. The figure shows 5 contact points. But in general it only takes 3 points to determine a circle. Clearly this circle is over-determined. But there are some restrictions. An obvious constraint is that the man is symmetric about the vertical dashed line in Figure 2. So we can discount points P_4 and P_5, since once we know P_1 and P_2, they are determined via reflection about the vertical diameter. So we have 3 points and the circle is determined.

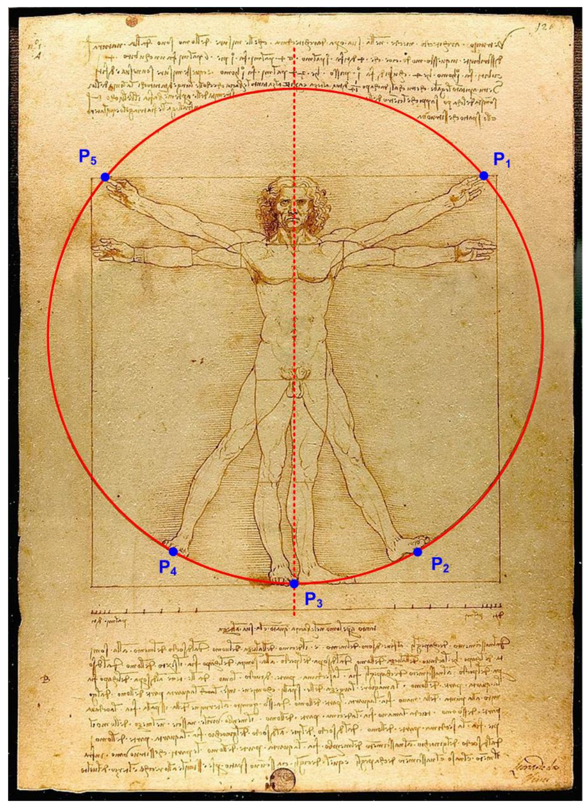

Figure 2 Vitruvian Man contact points on circle

Somehow that seemed to overlook some things. I felt the arms were somewhat arbitrary and that in fact fewer points determined this circle. So my question is:

Question. What is the minimum number of points needed to determine the Vitruvian Man circle? What are the additional characteristics of this circle?

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[1] Toby Lester is also the author of the wonderful book, *The Fourth Part of the World* (2009) [3], which ostensibly explains the origin of the 1507 Waldseemüller map of the world that included the name *America* for the first time on a map. In fact, Lester's book is a marvelous summary of the history of our understanding of the world and the universe from ancient times to the Age of Discovery. It approaches the subject through maps reflecting the knowledge of each age, with especial attention given to Ptolemy's 2nd century map in his *Geographia*. (Who knew Ptolemy introduced the idea of a latitude and longitude grid and that he had over 4000 latitude and longitude coordinates of geographic locations?) One of the most fascinating books I ever read.

Answer. Two. (P_3 and either P_1 or P_2)

Solution. My initial impression of the Vitruvian Man figure was misleading, since it seemed to suggest that the human body can wave its arms and legs and the extreme points will lie on a circle. Clearly the diagram shows this is not the case for the arms. So what is going on?

The point P_3 where the man is standing will always be on the circle (that is more or less the starting point). But there is an implicit constraint associated with P_3, namely, the axis of symmetry rises perpendicularly from it. This means the circle's center must lie along this line. Therefore, with one other point, the circle is determined. Figure 3 shows how this is done. Draw the line between P_3 and the second point P_2 and then construct the perpendicular bisector of this line (this is one of the first constructions a student performs in a plane geometry course). Extend the bisector until it intersects the symmetry axis through P_3. That point of intersection is the center of the desired circle (you might want to recall why this is so).

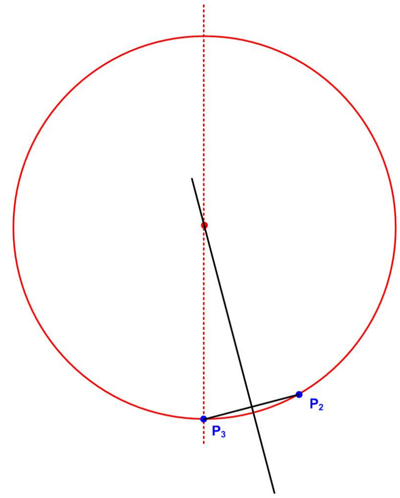

Figure 3 Finding the center of the circle.

So with the left raised leg in Figure 2 we have P_2 determined and therefore the circle. Look at Figure 4. It shows how the arm pivots about the shoulder socket and where its green circle of motion intersects the red circle determined by the foot, that becomes the point for the arm location.

Once we introduce the idea of a circle of motion for a limb relative to a socket, we realize that the leg involves the same circular motion around the (green) hip socket, which is not located in the (red)

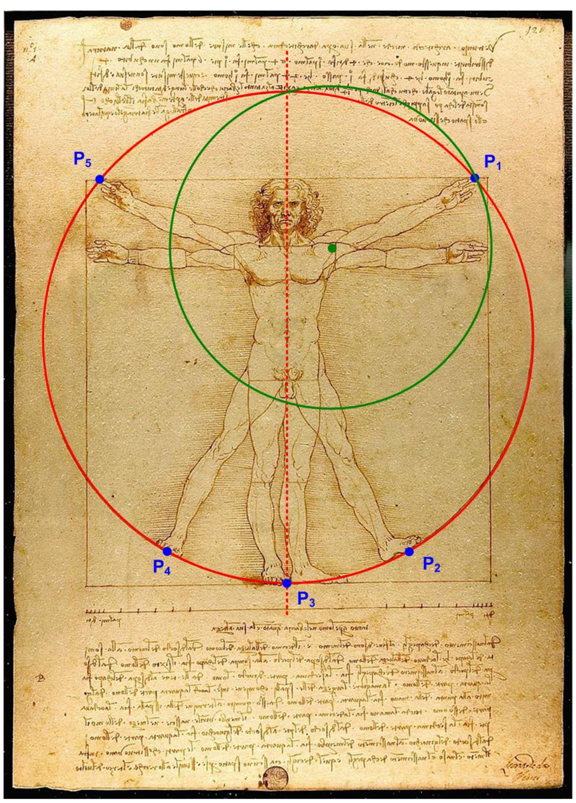

Figure 4 Arm circle

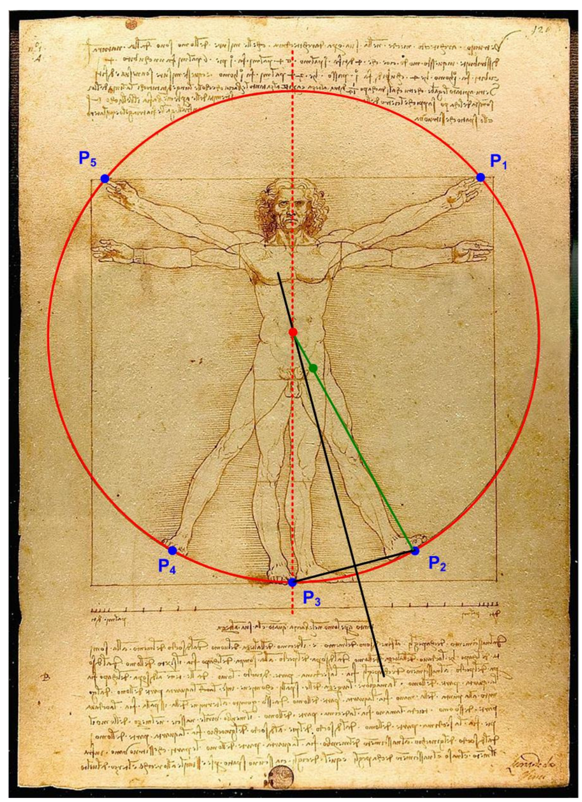

Figure 5 Off-center leg pivot (hip socket)

center of the Vitruvian Man circle. Figure 5 shows this situation. As the leg moves out, the radius of the red circle shrinks and will continue to intersect the green arm circle in Figure 4 so that we can continue to draw the Vitruvian Man circle.

But notice what happens when we move the leg back towards the vertical line of symmetry – the radius of the red circle grows and soon the red circle through the foot no longer intersects the green arm circle. So we would no longer be able to draw the Vitruvian Man circle in that case.

It is easy to see that, instead of P_2, we can take our second point at P_1 where the tip of the hand is, compute the corresponding radius of the (red) circle (still using P_3 and the line of symmetry), and then find the leg position that falls on the circle, *if it exists.* It should be clear that going this direction starting with the arm may be more problematical than starting with the leg.

It turns out, therefore, that a number of special circumstances are involved in the Vitruvian Man diagram and it is not as general as one might think initially.

The Square

In fact, there is even more to the diagram as I discovered after returning to read the rest of the book review. I had ignored the square in the figure, but that was essential to what Leonardo was showing, according to Lester, namely that the figure represented the integration of the two most fundamental geometric figures of a circle and a square with the body of man.

Figure 6 highlights the square in the Vitruvian Man drawing and indicates the four points of intersection with the standing man with arms outstretched. That the figure is a square follows from the assumption that the span of a man's arms equals his height. I had heard this notion when I was growing up, but I did not know where it originated or even whether it is true. It seems to be approximately correct, at least in my case. Whether it holds universally for all men or even women, I

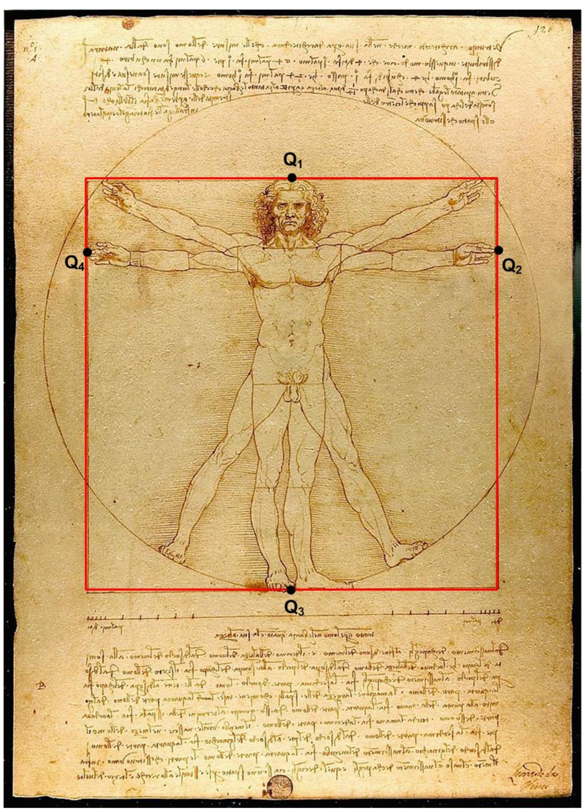

Figure 6 Man in square

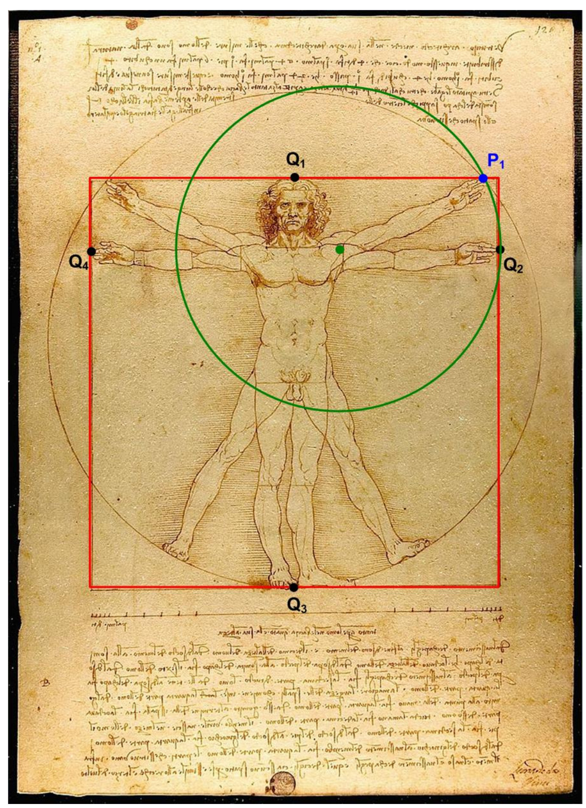

Figure 7 Selecting hand position

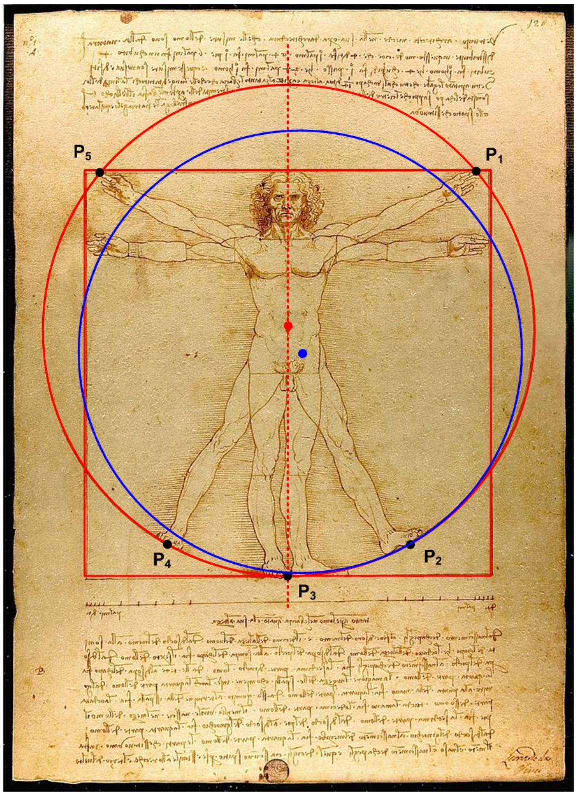

Figure 8 Selecting foot position P_2 on circle

don't know. But assuming this square relationship exists, then we have the hand position determined, namely, the point where the hand is aligned with the height of the man (Figure 7).

As I indicated in the discussion above, we can choose as our second point either the foot position or the hand position. (The first point is still where the man is standing, namely, Q_3 on the square or P_3 on the circle.) Now we have to ensure we can find a position for the foot on the (red) circle determined by the hand position. Figure 8 shows this is possible. In fact, due to the longer length of the leg the "leg" circle is larger and provides a wider effective region in which to choose a foot position (P_2).

Actually it is not critical that the rectangle be a square. All we are doing in the "square" case is selecting the hand position first and then the foot position. We would just have to ensure that any hand position at the height of the man will yield a possible foot position, and it looks like this will be the case.

References.

[1] Lopez, Jonathan, "The Measure of All Things," in *The New York Times* Sunday book review, 5 February 2012

[2] Lester, Toby, *Da Vinci's Ghost*: *Genius, Obsession, and How Leonardo Created the World in His Own Image*, Free Press, 2012

[3] Lester, Toby, *The Fourth Part of the World: The Race to the Ends of the Earth, and the Epic Story of the Map That Gave America Its Name*, Free Press, 2009