Two Circles Puzzle

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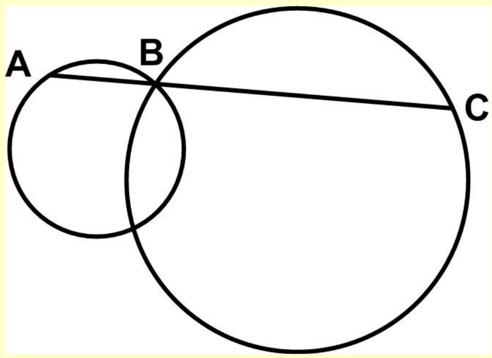
The following puzzle was found at the Futility Closet website:

(http://www.futilitycloset.com/2014/10/13/stretch-goals/, retrieved 6/20/2015)

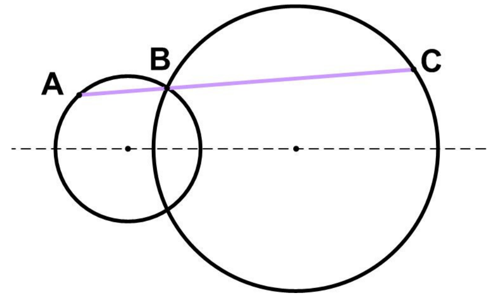
Stretch Goals (13 October 2014)



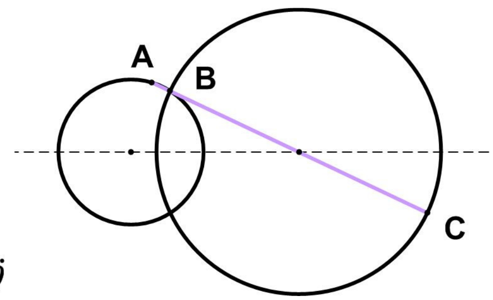
Two circles intersect. A line AC is drawn through one of the intersection points, B. AC can pivot around point B — what position will maximize its length?

I first worked out my solution and then I looked at theirs. Most of the Futility Closet problems have short, neat solutions, and the one they propose almost satisfies this characteristic. But I note that a fact they state is not so obvious (at least to me) and required proof, which made their solution almost as complicated as mine. Anyway, I will save this discussion for the next page in case the reader would like to think about the problem for themselves first.

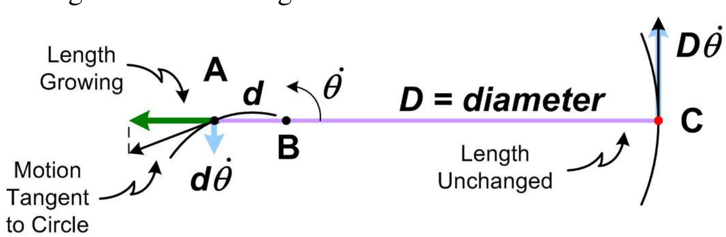
My Solution: The obvious approach would be to use calculus, but the equations would involve square roots and quadratics and be a nightmare. So we shall argue geometrically, then make an educated guess about the answer, and then prove the guess is correct.

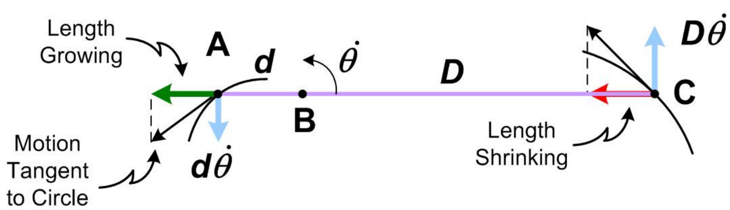


First rotate the original figure so that the line through the centers of the circles is horizontal. As a first guess, we might consider having the line pass through the center of the large circle so as to make the segment BC be the large circle's diameter, that is, the longest segment (chord) that will span a circle.



But notice that as we rotate the line about B, the segment BC hardly changes length (though it will shrink a bit), but the segment AB will grow rapidly, and thus make a longer overall line segment.

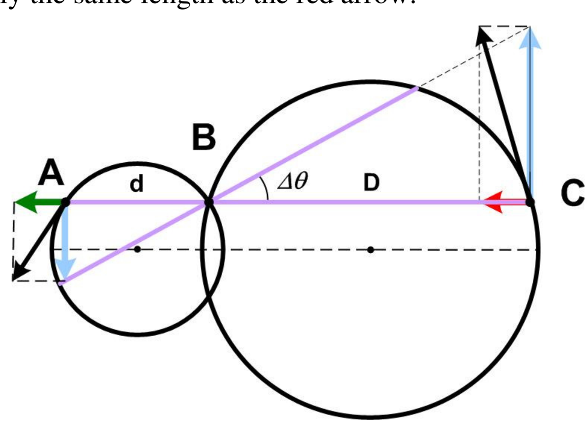




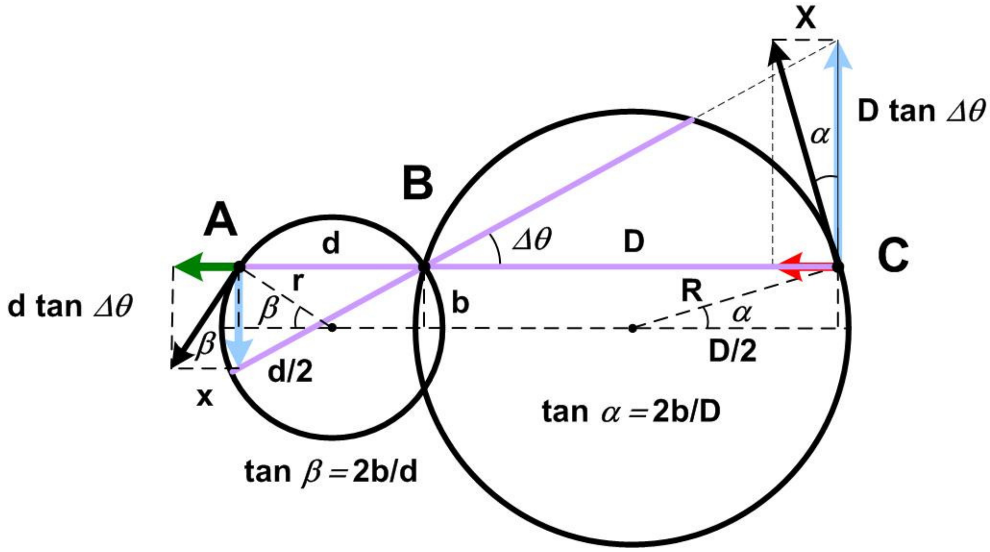
The dot over the angle theta represents the rotation rate, **D** is the length of the segment BC, and **d** is the length of the segment AB. The points A and C will move tangentially along their respective circles at a rate whose velocity component perpendicular to the line segment is equal to their distance from B times the rotation rate and whose velocity component along the line segment is the rate of growing or shrinking of the segment.

As C moves around the large circle in the counter clockwise direction, the segment BC (**D**) will shrink more and more rapidly. As A correspondingly moves around the smaller circle, the segment AB (**d**) will keep growing, but at a slower and slower amount (until **d** reaches the length of the diameter, at which point the segment AB stops growing and then begins to shrink). **So the point at which the segment AC is the longest, is when the growing at A exactly equals the shrinking at C.** In the diagrams this is when the green arrow is exactly the same length as the red arrow.

Now comes the guess. I played around with various line segments in Visio using the architecture length tool and discovered, somewhat surprisingly, that **the line AC parallel to the line through the centers of the circles seemed to yield the longest value**. So I sought to prove the instantaneous rate of growing and shrinking just balanced at that point.



Consider an infinitesimal increment $\Delta\theta$ in the rotation of the of the line segment about B from the horizontal. In the diagram a rather large increment is shown for purposes of visibility, but all of the geometry and discussion holds for smaller increments. The behavior of the rotation diagrams above is reflected accurately and to scale in the next figure. Now label the angles and line segments of the diagram as shown below, along with the trigonometric relationships. (It turns out that the radii are not needed, but they provide clarity. Also recall that a tangent to a circle is perpendicular to the radius of the circle drawn to the point of tangency.)



Then what we are trying to show is that X = x, that is, that the rate of shrinking of the segment at C is equal to the rate of growing of the segment at A. From the diagram we have

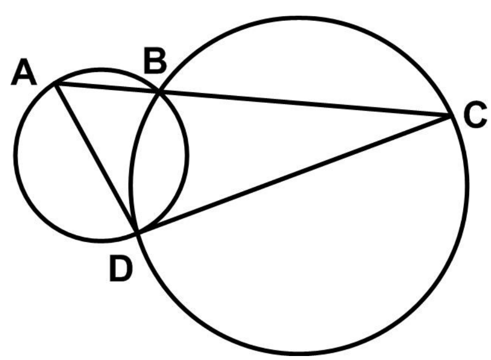
$$X = (\mathbf{D} \tan \Delta\theta)\,(\tan \alpha) = (\mathbf{D} \tan \Delta\theta)\,(2\,\mathbf{b}\,/\,\mathbf{D}) = 2\,\mathbf{b} \tan \Delta\theta$$

and

$$x = (\mathbf{d} \tan \Delta\theta)\,(\tan \beta) = (\mathbf{d} \tan \Delta\theta)\,(2\,\mathbf{b}\,/\,\mathbf{d}) = 2\,\mathbf{b} \tan \Delta\theta$$

and so they are indeed equal. The argument is highly dependent upon the fact that the line segment AC is parallel to the line through the centers of the circles. It should be clear from our discussion of shrinking and growing that there can only be one solution, and so the parallel line segment must be it.

Futility Closet Solution:



Call the other intersection point D and draw AD and DC. All angles inscribed in a circle and subtended by the same chord are equal, so angle BAD retains the same measure as A travels around its circle. Similarly, angle BCD remains the same as C travels around its circle. This means that triangle ADC will always have the same shape: As line AC pivots around B, triangle ADC will vary in size but remain self-similar.[1]

So which position will maximize its size? AD and DC are chords of their respective circles, and the longest chord is a diameter. So turn the triangle until both of these lines are diameters (this will happen

[1] JOS: If two triangles have two corresponding angles then same, then the third corresponding angle must also be the same, since the sum of the angles must always equal 180 degrees. So the triangles are similar.

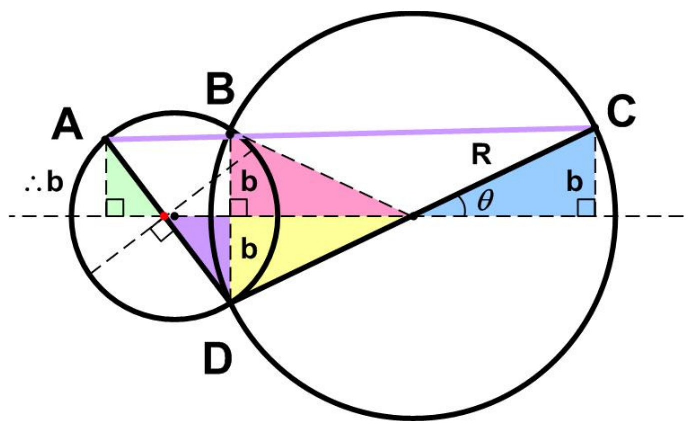
simultaneously[2]); at that point triangle ADC will reach its maximum size and line AC its maximum length.[3]

From Mogens Larsen in Richard Guy and Robert Woodrow, eds., *The Lighter Side of Mathematics*, 1994.

Remark 1: I need to show my solution agrees with the Futility Closet solution.

Clearly the Futility Closet solution is more elegant. What I need to show is that my solution agrees with theirs. In my last diagram above, extend the line from C through the center of the large circle to the opposite side at a point D. By congruent triangles D will be **b** units below the line through the centers of the circles and also **D**/2 units to left of the center of the large circle. This means D also lies on the small circle. The same argument for extending the line from A through the center of the small circle gives the same intersection point D on the small circle. This we have a triangle ADC with legs AD and DC being diameters of their respective circles, just as in the Futility Closet solution.

Remark 2: We need to show when the triangle ADC in the Futility Closet solution is maximal, both sides AD and DC are diameters of their respective circles.



Proof: There may be easier ways but my approach employs a number of congruent triangles. First assume DC is a diameter. Then the half segment of DC is a radius **R** of the larger circle.

Something we did not make explicit in the previous discussion, but the line through the centers of the circles is a perpendicular bisector of the line BD. So call the lengths of the half segment of BD **b.** Then the pink and yellow triangles are congruent (SSS). Drop the perpendicular from C to the line of centers. This line is parallel to the line BD. So the blue triangle is a right triangle with the same angles as the yellow triangle and a side in common (radius **R**) and therefore congruent to it. This means the perpendicular from C has the same length as **b**.

Thus the line AC is parallel to the line of centers (and thus must be the maximal line from our previous argument), which means the perpendicular from the point A to the line of centers is also **b** units long. Since this line is parallel to BD, the light green and lavender triangles are right triangles with a side in common (**b**) and the same angles. This means they are congruent.

Because the light green and lavender triangles are congruent, the red dot at their common vertex divides the line segment AD in half. Construct the perpendicular to AD at the red dot. Then the red dot is on the perpendicular bisector of AD and also on the perpendicular bisector of BD. The only point of intersection of perpendicular bisectors of chords on circles is the center. Therefore AD passes through the center of the small circle and must be a diameter.

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2 JOS: Why? This is key to the argument. If they were not both diameters, then setting leg DC to a diameter and then moving C a bit might allow A to lengthen the segment AC more (as in our discussion above) with both legs AD and DC being less than diameters. However, see my proof in Remark 2.

3 JOS: See Remark 3.

(I would be curious if there is an easier proof. There is! See Remark 4 below.)

Remark 3. What is "maximal"?

Actually we didn't really show that the "maximal" triangle has diameters for legs. We showed that if this particular triangle ADC with long side passing through B has DC for a diameter of the big circle, then the line AC through B is parallel to the line of centers and thus AD is a diameter for the small circle. The maximality came from my previous argument using the fact that AC is parallel to the line of centers. But we could get it from the similar triangle argument.

My discussion makes the maximal length of AC clear. The similar triangle argument may not be as certain. How do we know there can't be another triangle with the sides not diameters that has longer third side?

In fact, what the Futility Closet solution is saying is that we are not looking at arbitrary triangles in the two circles but only those with a side through B and vertex at D, making the sides AD and DC chords of their respective circles. It was shown that all such triangles must be similar to one another. Therefore the ratio of the side AC in the "maximal" triangle (that is, the triangle with diameters for legs) to the side AC in any similar triangle must be the same as the ratio of a diameter leg, say DC, to the corresponding leg of the similar triangle. Since the DC ratio must be greater than 1 (since DC being a diameter is the largest chord), then so must the ratio of the side AC to its corresponding side be greater than 1, thus making it of maximal length. This is all probably obvious, but there needed to be a little thought to make sure.

Remark 4. Easier Futility Closet Solution explanation

Using the argument in the previous paragraph, we can go directly from the similar triangle argument to the answer.

Among all the similar triangles (which capture all the instances of interest for line AC through B), consider the triangle with side DC through the center of the large circle and thus a diameter. Then DC is greater than the length of the corresponding leg in all the other similar triangles, *as is chord DA*. This means DA must be the diameter of the small circle, but that is not needed for the argument that AC is greater than the length of the corresponding leg in the other similar triangles. *So the **two** legs of the triangle being diameters simultaneously is a red herring*.