Slow Lane Problem

28 July 2013, rev 15 January 2017

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Paul Krugman in his 26 July 2013 *New York Times* column "The Conscience of a Liberal" provided the following analysis about always being caught in the slow lane of a congested highway:

> **Friday Night Music: Sprawl, Again** [1]
>
> Consider the following thought experiment: you are driving on a road — let's arbitrarily call it Interstate 91 — and must choose a lane. Traffic is so heavy that you can't really change lanes thereafter. But there are many bad patches along the road; half of the distance can be covered at 60 miles an hour, but the other half only at 15.
>
> You might imagine that your average speed is halfway between 15 and 60, but a little thought shows that this isn't true: your average speed is only 24 miles an hour. Also, the lanes aren't perfectly correlated: sometimes your lane is going 60 while the next is going 15, sometimes it's the reverse. Again, you might think that this means you spend equal amounts of time watching the other lane whiz by and whizzing by yourself, but not so: you spend four times as much time watching the other guys race past.
>
> And this creates intense frustration and anger, a sense that it's grossly unfair that you are in the wrong lane. This sense persists even though (a) you have worked out the analysis above, and realize that in principle the lanes are equally good or bad and (b) you have in fact been playing leapfrog with the same Boltbus the whole way, so that you know that in fact neither lane is better. No matter; you are angry, frazzled, and late for your family event …

Krugman offered a further explanation in a later post that we will consider in a moment, but it did not explain why the average speed was 24 mph.

Question 1: Why is the average speed of 60 mph and 15 mph in the problem just 24 mph?

Solution

The key to the solution is to consider the unknown times. Note that we also do not know the distance traveled, so designate this unknown by d. Let t_F and t_S be the times it takes to drive half the distance d at 60 mph and 15 mph, respectively. That is,

$$\frac{d/2}{t_F} = 60 \text{ mph} \quad \text{and} \quad \frac{d/2}{t_S} = 15 \text{ mph} \tag{1}$$

The total time t to drive distance d is $t = t_F + t_S$. So the average speed to drive d is

$$\frac{d}{t} = \frac{d}{t_F + t_S} = \frac{d}{\frac{d}{120} + \frac{d}{30}} = \frac{120}{5} = \boxed{24 \text{ mph}}$$

It's rather neat how the unknown distance cancels out.

This situation relates to an old chestnut in a number of puzzle books. For example,

[1] http://krugman.blogs.nytimes.com/2013/07/26/friday-night-music-sprawl-again/, July 26, 2013, 9:21 pm

No Radar Trap. I drive an average speed of 30 miles per hour to the railroad station each morning and just catch my train. On a particular morning there was a lot of traffic and at the halfway point I found I had averaged only 15 miles per hour. How fast must I drive the rest of the way to catch my train? [2]

Solution

Again the key is to consider the times. If the distance to the railroad station is the unknown d and the time it takes to get there at 30 mph is the unknown t, then we have $d/t = 30$. So the time it takes to get to the train is $t = d/30$ hours. Let t_0 be the time it takes to drive half the distance at 15 mph. Then $(d/2)/t_0 = 15$. So the average speed needed for the other half of the distance is $r = (d/2)/(t - t_0)$. But t_0 already equals $d/30$, which is the total time available to get to the train, namely t. So the remaining average speed would have to be infinite. That is, it is impossible to reach the train on time at any speed.

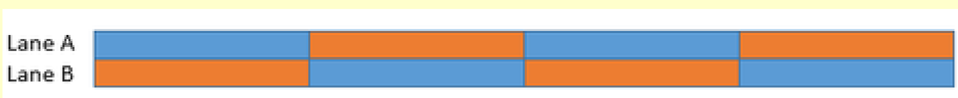
Question 2: Why does it appear you always spend more time in the slower lane than other drivers?

Krugman explains:

Life In The Slow Lane (Trivial) [3]

I see that quite a few readers think the math in my last Friday Night Music post was wrong. Since whining about traffic is very important, I guess I need to set everyone straight.

So, let's be concrete (which is better than asphalt). Imagine that our journey is 120 miles. Half of the 120 miles is good distance, which you can cover at 60 miles an hour; the other half is bad distance, which you cover at only 15 miles an hour. In the figure below, I assume that good and bad stretches — indicated by blue and red respectively — come in 30-mile blocks, and are perfectly uncorrelated:



If you think about if for a minute, you'll see the following:

1. It will take just 1/2 hour to traverse each blue block, but it will take 2 hours to traverse each red block. In total, therefore, although half the distance is red, half blue, you will spend only 1 hour in the blue and 4 hours in the red.
2. Whenever you are in a blue block, you will be whizzing past the other lane. Whenever you are in a red block, the other lane will be whizzing past you.
3. Therefore, you will spend 4 times as much time watching the other lane whiz past as you will being the one doing the whizzing. …

Sometimes there is a little hesitation in believing a general result when only a specific example is provided. Perhaps tweaking Krugman's explanation a bit will show it holds in general.

Suppose first that the speeds in the lanes were correlated and collect all the 60 mph intervals to the left and all the 15 mph intervals to the right:

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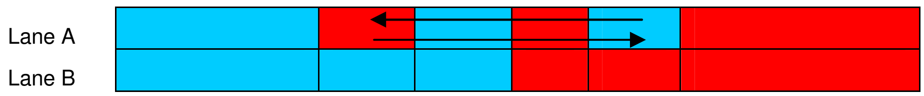
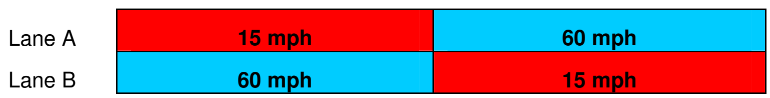
[2] #17 "No Radar Trap" in L. H. Longley-Cook, *Work This One Out*, 1960, Crest reprint 1962.

[3] http://krugman.blogs.nytimes.com/2013/07/28/life-in-the-slow-lane-trivial/, July 28, 2013, 6:37 am

Now suppose you are driving in Lane A in a 15 mph interval whereas the cars in Lane B are traveling at 60 mph. Then we would have the following picture:



Notice the swapped intervals are the same size. So collecting all the instances where the traffic in Lane A and Lane B is moving at different speeds, we have:



where again the swapped intervals and thus different-speed intervals are the same size and half of the total. (The total distance may be less than the original distance if there are correlated intervals of the same speed.)

So for you driving in Lane A the situation is the same as for Question 1. And Equation (1) shows the ratio of time you spend in the slow interval is 4 times as long as the time spent in the fast interval. And therefore you are watching faster cars go by in Lane B four times longer than you are going fast in Lane A yourself.