Kepler's Laws and Newton's Laws

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Years ago (1963) I got the paperback *The Calculus:A Genetic Approach*, by Otto Toeplitz [1], which presented the basic ideas of the differential and integral calculus from a historical point of view. At the time I found the presentation "messy" and difficult to follow. Yet over the years I kept returning to the book as I learned more mathematics and become more interested in the history of mathematics. I kept thinking that some of the modern-day abstractions might be more easily understood if I returned to their origins and tried to follow the more concrete and perhaps intuitive approach of their creators.

Alas, I discovered some more fundamental truisms. Namely, that the initial creators or modifiers of ideas had a confused notion of what these ideas were, or rather that they were approaching these ideas from a much different direction from today. They were usually exploring detailed issues or examples of a more advanced nature and were only getting glimmerings of underlying simplifying abstractions. For example, today we spend a lot of time in the beginning of calculus defining what a function is. Historically, today's definition was arrived at relatively recently (end of 19th century) based on the evolving idea that a function could be represented by a Fourier series. The limits of Fourier series often had very strange properties that challenged earlier ideas of what a function should be. The coefficients of the Fourier series involved integrals of this limit function, which consequently raised questions about what integration should be. The study of Fourier series also spawned the ideas of point set topology and Cantor's great achievement of categorizing infinite sets. Now we discuss functions (and these other concepts) way before we reach the study of a Fourier series, and so Fourier series would not be a good motivator for an elementary class.[1]

But one thing Toeplitz did at the end of his book that I had not seen in other texts was to show the equivalence of Kepler's Laws and Newton's Law of Gravity.[2] I thought I would try to emulate his approach with more modern notation (vectors) and arguments in hopes of extracting the essential ideas from the clutter.

To keep things as clear as possible I thought I would state all the laws up front. I used my old college physics text book for the statements ([2]), since Toeplitz had a slightly simplifying approach that initially omitted the idea of mass and forces. Toeplitz also deviated from more common usage by reversing the order of Kepler's Laws (following Newton and Kepler himself, actually). I have used the more traditional order.[3]

Kepler's Laws ([2] p. 42)

1. The orbit of each planet is an ellipse with the sun at one focus.
2. The radius vector drawn from the sun to a planet sweeps out equal areas in equal times (*Law of Areas*).
3. The squares of the periods of revolution are proportional to the cubes of the semimajor axes of the elliptical orbits. (*Harmonic Law*)

Newton's Laws of Motion ([2] p. 30)

1. Every body continues in its state of rest or of uniform motion in a straight line, except in so far as it is compelled by forces to change that state. (*Law of Inertia*)

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1 David Bressoud recently published an analysis text (2007) that takes this very historical approach ([4]).
2 Since 1963 Bressoud has developed this theme in his excellent 1991 text ([3]).
3 For more historical background to the evolution of these ideas from Kepler to Newton, see Cohen's book ([5]).

2. Time rate of change of "quality of motion" (i.e., rate of change of momentum = *ma*) is proportional to force and has the direction of the force (**F** = *m***a**)
3. To every action there is always an equal and contrary reaction: or the mutual actions of any two bodies are always equal and oppositely directed along the same straight line.

Newton's Universal Law of Gravity ([2] p. 41)

Every particle in the universe attracts every other particle [along the line between them] with a force that is proportional to the product of the masses and inversely proportional to the square of the distance between the particles. ($F=GMm/r^2$)

Newton's Law (Acceleration Version) (Toeplitz, [1] p. 156)

The acceleration of a planet is always directed toward the sun, and its magnitude is inversely proportional to the square of its distance from the sun.

Toeplitz does not mention forces or masses in his version, at least initially. A priori the constant of proportionality may depend on properties of the planet and of the sun, that is, different planets at the same distance from the sun may have different accelerations, and similarly for a planet at the same distance from different suns. But for the same planet and the same sun, the acceleration only depends on the distance.

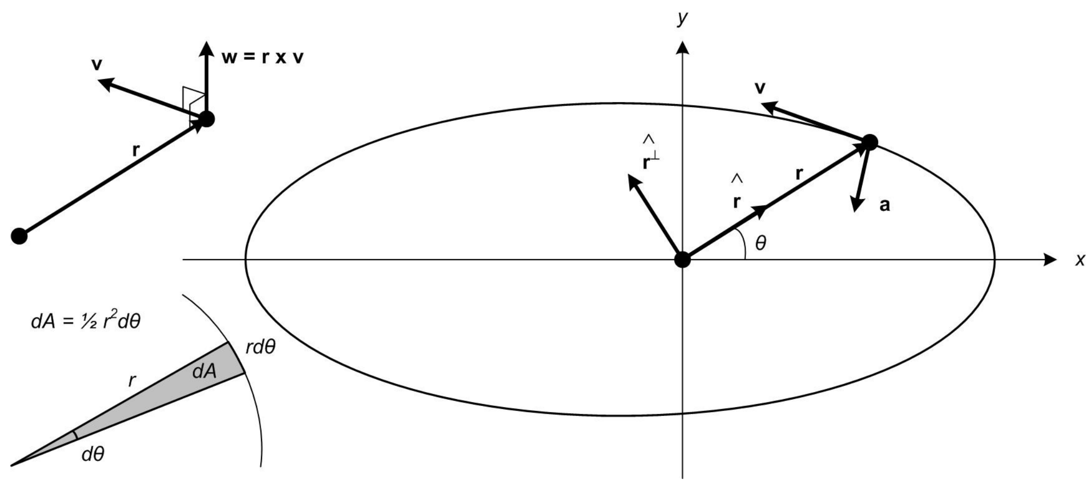


Figure 1 Vector and Area Relationships

Planar Motion Theorem If a particle moves so that its acceleration is always along the line joining another particle, then it describes a planar curve.

Proof. Using the diagrams in Figure 1, let $\mathbf{w} = \mathbf{r} \times \mathbf{v}$, where all vectors are now assumed to be in 3-space. Let $\hat{\mathbf{w}}$ represent the unit vector. Then $\hat{\mathbf{w}}$ is perpendicular to both **r** and **v**, and **r** moves in a plane if and only if $\hat{\mathbf{w}}$ does not change over time, that is, only moves parallel to itself. This is equivalent to saying

$$\frac{d}{dt}\hat{\mathbf{w}} = \dot{\hat{\mathbf{w}}} = \mathbf{0}$$

Now letting $\mathbf{a} = \dot{\mathbf{v}}$ represent the acceleration vector and $w = |\mathbf{w}|$ represent the length of a vector,

$$\dot{\hat{\mathbf{w}}} = \frac{1}{w}(\mathbf{r}\times\mathbf{a}) - \hat{\mathbf{w}}(\hat{\mathbf{w}} \bullet (\mathbf{r}\times\mathbf{a}))$$

$$\therefore \mathbf{r} \text{ parallel to } \mathbf{a} \Rightarrow \mathbf{r}\times\mathbf{a} = \mathbf{0} \Rightarrow \dot{\hat{\mathbf{w}}} = \mathbf{0}$$

And so if a particle always moves so that its acceleration is along the line joining another particle, that is, so that **r** is parallel to **a**, then the curve of motion is planar.

Equal Areas Theorem (cf. Toeplitz, [1] p.152) A particle moves so that its acceleration is always directed along the line joining another particle if and only if the particle moves in a planar curve and the line between the two particles sweeps out equal areas in equal amounts of time.

Proof. We have already shown the that radial acceleration implies planar motion (Planar Motion Theorem). So it suffices to show for a planar curve that radial acceleration is equivalent to the Law of Areas. Using the diagrams above in Figure 1, we have for the area A swept out by the radial vector from one particle to the other the following time rate of change derivatives

$$\dot{A} = \tfrac{1}{2} r^2 \dot{\theta} \quad \text{implies} \quad \ddot{A} = r\dot{r}\dot{\theta} + \tfrac{1}{2} r^2 \ddot{\theta}$$

Thus the Law of Areas means $\dot{A} = \text{constant}$ or $\ddot{A} = 0$.

Let $\hat{\mathbf{r}}$ represent the unit vector along $\mathbf{r}$ and $\hat{\mathbf{r}}^{\perp}$ the counterclockwise unit vector perpendicular to $\mathbf{r}$ in the plane (see Figure 1), namely,

$$\hat{\mathbf{r}} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta \quad \text{and} \quad \hat{\mathbf{r}}^{\perp} = -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta \quad \text{so that}$$

$$\dot{\hat{\mathbf{r}}} = \hat{\mathbf{r}}^{\perp}\dot{\theta} \quad \text{and} \quad \dot{\hat{\mathbf{r}}}^{\perp} = -\hat{\mathbf{r}}\dot{\theta}$$

Now

$$\mathbf{v} = \dot{\mathbf{r}} = \hat{\mathbf{r}}\dot{r} + \hat{\mathbf{r}}^{\perp} r\dot{\theta} \quad \text{and}$$

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{r}}^{\perp} \quad \text{so that}$$

$$\boxed{\mathbf{a} = (\ddot{r} - \frac{4\dot{A}^2}{r^3})\hat{\mathbf{r}} + \frac{2\ddot{A}}{r}\hat{\mathbf{r}}^{\perp}} \tag{1}$$

which implies $\mathbf{a} \parallel \mathbf{r} \Leftrightarrow \ddot{A} = 0$. QED

Note: these last two theorems only involved acceleration *along* the line between the particles and not actually acceleration of one particle *towards* the other. That is, we only used $\mathbf{a} \parallel \mathbf{r}$ with no mention of sign.

Kepler's First Two Laws ⇒ Newton's Law (Acceleration version)

Proof: We could use the Equal Areas Theorem to get the radial acceleration, but we need more. We need to show the planet's acceleration is toward the sun and that it is inversely proportional to the

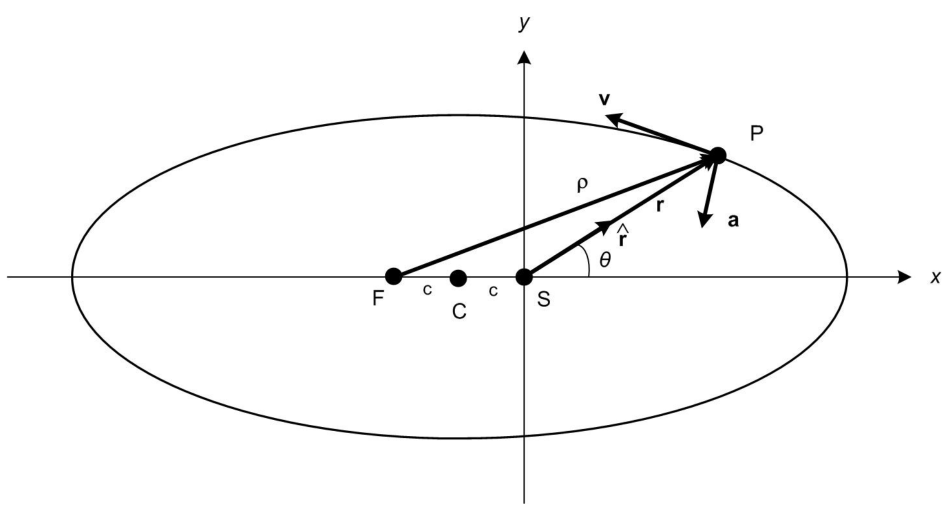


Figure 2 Elliptical Planetary Orbit

square of the distance.

Assumption 1 Kepler's First Law (Elliptical Path)

Figure 2 represents the planet at P moving along the ellipse with the sun at one focus S. F represents the other focus of the ellipse with a vector $\boldsymbol{\rho}$ from F to P. Then the path generated by P is an ellipse if and only if the sum of the distances from P to the foci is a constant, that is, the path is an ellipse $\Leftrightarrow \rho + r = 2a$ for all θ and some constant a (= semimajor axis). (Semiminor axis b satisfies $b^2 + c^2 = a^2$) Therefore,

$$\rho^2 = (2a - r)^2 = 4a^2 - 4ar + r^2$$

From vectors

$$\rho^2 = \boldsymbol{\rho}\cdot\boldsymbol{\rho} = (2c\,\mathbf{i} + \mathbf{r}) \cdot (2c\,\mathbf{i} + \mathbf{r}) = 4c^2 - 4c\,\mathbf{r}\cdot\mathbf{i} + r^2$$

Combining equations gives

$$ar + c\,\mathbf{r}\cdot\mathbf{i} = a^2 - c^2 \;\Rightarrow\; r = a - c > 0 \text{ when } \theta = 0 \text{ for an ellipse (so that } \mathbf{r}\cdot\mathbf{i} = r\text{).}$$

Defining $e = c/a < 1$ (eccentricity) and $p = a(1 - e^2) > 0$ ($p = b^2/a$ the semi-latus rectum), we have

$$r + e\,\mathbf{r}\cdot\mathbf{i} = p \tag{2}$$

Setting $\mathbf{r} = r\hat{\mathbf{r}}$, we get $r(1 + e\hat{\mathbf{r}}\cdot\hat{\mathbf{i}}) = p$, so that

$$r = \frac{p}{1 + e\cos\theta}$$

Differentiating equation (2) twice we have $\ddot{r} + e\ddot{\mathbf{r}}\cdot\hat{\mathbf{i}} = 0$. Using $\ddot{\mathbf{r}}$ for $\mathbf{a}$ in equation (1) and **Assumption 2** Kepler's Second Law ($\ddot{A} = 0$, $\dot{A} =$ constant),

$$0 = \ddot{r} + e(\ddot{r} - \frac{4\dot{A}^2}{r^3})\hat{\mathbf{r}}\cdot\hat{\mathbf{i}}$$

From (2) with $\mathbf{r} = r\hat{\mathbf{r}}$, we have for $\dot{A} =$ constant and the path an ellipse

$$0 = \ddot{r} + (\ddot{r} - \frac{4\dot{A}^2}{r^3})(\frac{p}{r} - 1) = (\ddot{r} - \frac{4\dot{A}^2}{r^3})\frac{p}{r} + \frac{4\dot{A}^2}{r^3} \;\Rightarrow$$

$$\ddot{r} - \frac{4\dot{A}^2}{r^3} = -\frac{\gamma}{r^2} \tag{3}$$

where $\gamma = \frac{4\dot{A}^2}{p} = \frac{4\dot{A}^2}{a(1-e^2)} > 0$ and is constant over time. And so from equation (1) the acceleration of P is

$$\mathbf{a} = -\frac{\gamma}{r^2}\hat{\mathbf{r}},\;\; \gamma > 0.$$

Therefore the acceleration of the planet P is towards the sun S and inversely proportional to the square of the distance between them. QED

Newton's Law (Acceleration version) $\Rightarrow$ **Kepler's First Two Laws**

Proof: **Assumption 3** Newton's Law (Acceleration version) ($\mathbf{a} = -\frac{\gamma}{r^2}\hat{\mathbf{r}}$, $\gamma > 0$)

By the Equal Areas Theorem, the planet moves along a planar curve and the line from it to the sun sweeps out equal areas in equal times (Kepler's Second Law: Law of Areas). It remains to show the planar curve is an ellipse (Kepler's First Law). From equation (1), the Law of Areas, and Newton's Law (Acceleration version), we get (cf. equation (3))

$$\ddot{r} = \frac{4\dot{A}^2}{r^3} - \frac{\gamma}{r^2} \tag{4}$$

Through a series of transformations, Toeplitz converts the derivatives of r with respect to time to those with respect to θ

$$\ddot{r} = -\frac{4\dot{A}^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

so that equation (4) becomes

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{1}{r} + \frac{\gamma}{4\dot{A}^2} \Rightarrow \frac{d^2u}{d\theta^2} = -u \text{ where } u = \frac{1}{r} - \frac{\gamma}{4\dot{A}^2},$$

with solution $u = D\cos(\theta - E) = D\cos\theta$ after a rotation of axes that eliminates E. Then

$$\frac{1}{r} = \frac{\gamma}{4\dot{A}^2} + D\cos\theta \Rightarrow r = \frac{1}{\gamma/4\dot{A}^2 + D\cos\theta} = \frac{p}{1 + e\cos\theta} \tag{5}$$

where $p = \frac{4\dot{A}^2}{\gamma}$ and $e = pD = \frac{4\dot{A}^2}{\gamma}D$. Thus equation (5) defines a conic section and is an ellipse if D is such that $e < 1$, a hyperbola if D is such that $e > 1$, and a parabola if D is such that $e = 1$.

Note: this implies a hyperbola and a parabola also satisfy Newton's Law. In fact, going the other way, we could argue as in the "Kepler's Laws ⇒ Newton's Law" case using a hyperbola ($p = -a(1 - e^2) > 0$) or parabola ($p = 2d > 0$ where d is the distance from focus to vertex) instead of an ellipse to arrive at the inverse square law. So there is not a strict equivalence between Newton's Law (Acceleration version) and Kepler's first two laws, but rather with a more general version of Kepler's First Law if we allow all conic sections for orbits.

Assume Kepler's first two Laws, then

Kepler's Third Law ⇔ γ constant for all planets

Proof: Assuming Kepler's first two laws (in particular, the path is an ellipse), the line joining a planet to the sun sweeps our the entire area of the ellipse in a period T of one revolution. Thus for an ellipse with semimajor axis a and semiminor axis $b = a\sqrt{1-e^2}$ and corresponding area $ab\pi$, the constant area rate of change $\dot{A}$ is

$$\dot{A} = \frac{ab\pi}{T} \Rightarrow \gamma = \frac{4\dot{A}^2}{p} = \frac{4a^2b^2\pi^2}{a(1-e^2)T^2} = 4\pi^2\frac{a^3}{T^2}$$

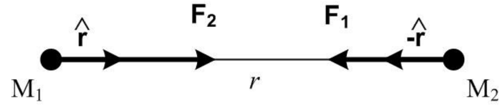
Therefore Kepler's Third Law ($a^3 \propto T^2$ for all planets around the sun) ⇔ γ is constant for all planets. QED

Note: though γ is now shown to be independent of the planets, it still depends on the sun. So if we look at say the similar system of the moons of Jupiter revolving around Jupiter, they satisfy a similar inverse square law but with a different constant γ whose value depends on Jupiter.

Assume Kepler's three Laws and Newton's 2d Law of Motion, then

Newton's 3d Law of Motion ⇔ every γ is proportional to its ("sun") mass.

Proof: Newton assumed the force of gravity was mutual and so for a pair of objects each one could assume the role of sun and the other the planet. This means the force exerted by the first on the second is oppositely directed along the line between them to the force exerted by the second on the first.



Setting the unit vector $\hat{\mathbf{r}}$ arbitrarily from the first object toward the second, from Newton's Law (Acceleration version) we have the following accelerations

$$\mathbf{a}_2 = -\frac{\gamma_1}{r^2}\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{a}_1 = \frac{\gamma_2}{r^2}\hat{\mathbf{r}}$$

of the second and first objects, respectively, caused by the other object (acting as "sun"), where the dependencies of the γ's on the "sun" object in each case are made explicit. By Newton's Second Law of Motion we have the corresponding expressions for the forces

$$\mathbf{F}_1 = M_2\mathbf{a}_2 = -\frac{\gamma_1 M_2}{r^2}\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{F}_2 = M_1\mathbf{a}_1 = \frac{\gamma_2 M_1}{r^2}\hat{\mathbf{r}}$$

Therefore,

$$\mathbf{F}_1 = -\mathbf{F}_2 \iff \gamma_1 M_2 = \gamma_2 M_1 \iff \frac{\gamma_1}{M_1} = \frac{\gamma_2}{M_2} = G, \text{ a constant.}$$

Thus the forces are equal and opposite if and only if the γ's are proportional to the masses of their respective objects ("suns"). QED

Thus the magnitude of the force of gravity between two objects with masses M_1 and M_2 separated by a distance r is given by

$$F = G\frac{M_1 M_2}{r^2} \quad \left(\text{since } \mathbf{F}_1 = M_2\mathbf{a}_2 = -\frac{\gamma_1 M_2}{r^2}\hat{\mathbf{r}} = -\boxed{\frac{GM_1M_2}{r^2}}\hat{\mathbf{r}} = -\frac{\gamma_2 M_1}{r^2}\hat{\mathbf{r}} = -M_1\mathbf{a}_1 = -\mathbf{F}_2\right)$$

where G is a constant independent of the masses. This constant of proportionality G is Newton's Universal Gravitation constant. Therefore we have

Assume Newton's Laws of Motion, then

Kepler's Laws ⇔ Newton's Universal Law of Gravity

Note: A similar discussion as the above is given in Bressoud [3], Section 3.3 Orbital Mechanics. In fact Bressoud has a nice, completely vector approach to "Newton's Law (Acceleration version) ⇒ Kepler's First Two Laws" which avoids solving differential equations.

Significance

In the above discussion I have tried to flag at each juncture just what specific assumptions apply at that moment. This enables us to see some internal equivalences. Basically, radial acceleration of an object is equivalent to equal areas swept out in equal times. And an object's path being a conic section is equivalent to the radial acceleration being toward the sun and satisfying the inverse square law. The harmonic law is equivalent to the inverse square law being independent of the object. And then amazingly, the innocuous 3d Law of Newton's Laws of Motion (every action has an equal and opposite reaction) is equivalent to there being a universal constant of gravity G that does not depend on the sun, planets, or any objects with mass attracting one another. This is essentially where the idea of the apple falling to the earth is subject to the same law as the planets falling towards the sun.

But there is something even more profound in all this. All the laws discussed above had mathematical representations, even though they were referring to physical situations. And the derivations of the relationships between these laws, including the equivalences, were all done solely with mathematics and without any physical reasoning. The physical phenomena that Kepler observed and codified mathematically in his laws turn out somehow to be inherent in the mathematical properties of Newton's laws (of motion and of gravity), which also represent physical phenomena (see Figure 3).

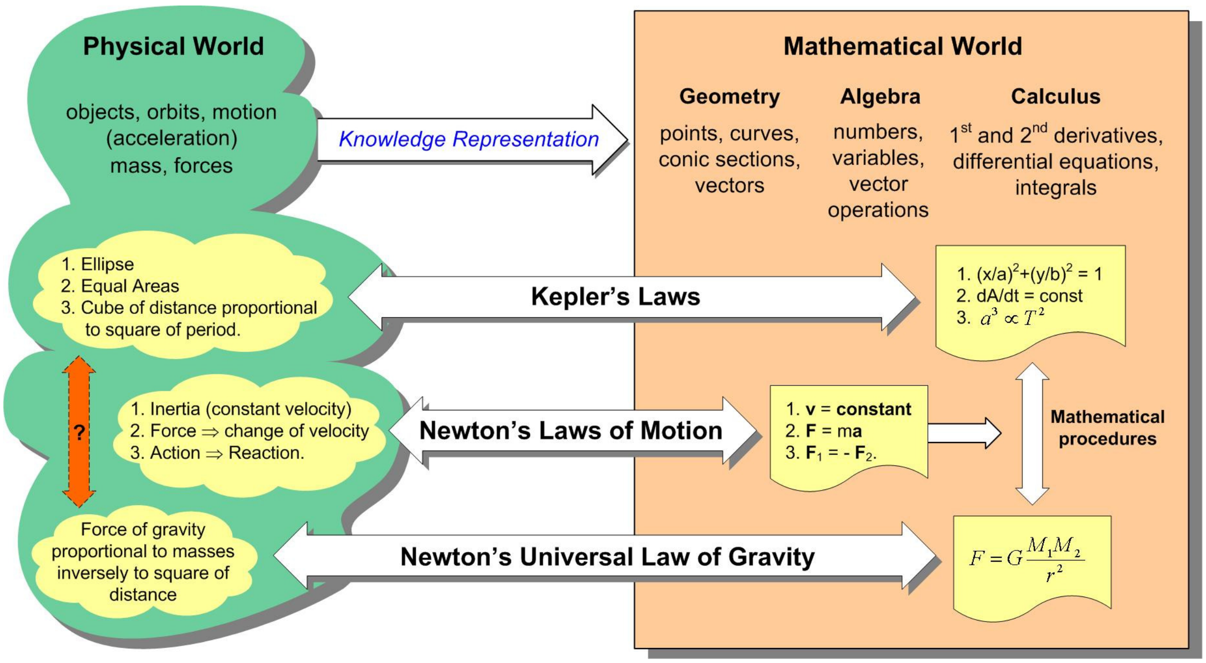


Figure 3 Mathematical Representation of Physical Reality

I remember when I worked at the Geophysical Fluid Dynamics Laboratory there was an oceanographer who wrote down a particular form of the Navier-Stokes equations that represented the fluid motion of certain ocean regions. He manipulated the equations and produced successive transformations into alternate forms. When I asked him to show me the mathematical derivation, he admitted he did it "physically," that is, each term in the original equation had a physical meaning to him and he knew physically how these terms transformed into other terms that represented physical phenomena. In other words, to him the partial derivatives and other mathematical symbols in the equations were just labels or names of physical entities and he moved from one physical entity to the next using physical reasoning.

The derivation above of the equivalence of Kepler's Laws and Newton's Law of Gravitation was not like that. It was all done mathematically. There was no physical mechanism indicated that would show how these two Laws were equivalent, only mathematical manipulations. That is the great mystery of mathematics and physical reality.

And that is the great revolution spawned by Newton and his laws. After Newton, mathematics became the primary tool for describing nature and for "explaining" it. That is, predictions and relationships between physical phenomena were all eventually captured in mathematical structures and became subject to mathematical rules, not physical ones. Often, like with Kepler, physical phenomena would be observed and crude mathematical notions applied, which often did not quite fit into current mathematical thinking. But soon the mathematicians worked out a mathematical structure with its own logic to organize these new ideas. Such was the case with Dirac's delta function, which was zero everywhere except at the origin, where it was sufficiently "infinite" so that its integral over the whole real line was 1. Schwartz' theory of distributions was developed to make sense out of this "function."

More often, however, the order of precedence is reversed, that is, new physical phenomena seem to occur and soon a previously created mathematical concept turns out to be applicable. An example is Heisenberg's formulation of quantum mechanics in which he expressed quantum notions with arrays of elements and defined a type of multiplication of these arrays. Soon someone told him he was actually using the theory of matrices invented by the mathematician Cayley decades before.[4] Einstein discovered that he needed to consider his spacetime to be a curved geometry and his mathematical friends pointed him to the work of Riemann in the previous century developing the theory of manifolds and the recent work of Levi-Civita on differential geometry.[5] Of course, there is the amazing wedding of group theory developed by Lagrange, Galois, and Lie to modern quantum mechanics and the discovery of new particles.

Ultimately we do not understand why mathematics should permeate physical reality so thoroughly. What makes it even more mysterious, as suggested in the previous paragraph, is that many of the mathematical concepts "explaining" physical reality were created out of the minds of mathematicians, springing from their imagination like some sort of game or artistic creation without any thought of physical meaning.

This mysterious and almost mystical dominion of mathematics in every corner of modern science all started with Newton and his amazing mathematical derivation of Kepler's Laws from his Laws of Motion and Law of Gravity.

References:

[1] Toeplitz, Otto, *The Calculus: A Genetic Approach*, University of Chicago Press, 1963 (from 1949 German version), reprint 2007 with Forward by David Bressoud. Fascinating historical presentation of the basic ideas of the differential and integral calculus with infinite series. Because things are always confused initially when investigators are not clear about what they are discovering, the approach appears a bit messy and may require some mathematical sophistication to understand. Still it is quite revealing and rewarding.

[2] Ference, Jr., Michael, Harvey B. Lemon, and Reginald J. Stephenson, *Analytical Experimental Physics*, 2d rev edition, University of Chicago Press, Chicago, Illinois, 1956. Rather succinct and extensive compendium of basic physics compared with more modern approaches. (Out of print, though can be found used at Amazon, for example. Not really needed, however.)

[3] Bressoud, David M., *Second Year Calculus: From Celestial Mechanics to Special Relativity,* Springer-Verlag, New York, 1991. Includes a good and thorough technical presentation of the ideas introduced in my discussion above and more.

[4] Bressoud, David M., *A Radical Approach to Real Analysis*, 2nd edition, Mathematical Association of America, 2007. Quasi-historical, "genetic" approach to analysis (following a second year calculus course) that begins with Fourier Series in the early 1800s and develops the notions of

[4] See Hoffmann ([6]) for a basic history of early quantum mechanics.

[5] See Gardner ([7]) for an accessible explanation of special and general relativity, and Steinmetz ([8]) for some more details.

modern analysis through the issues raised by the impact on classical mathematics of these fundamental series.

[5] Cohen, I. Bernard, *The Birth of a New Physics* (Revised and Updated), W. W. Norton & Company, 1985. Good historical survey of the issues in celestial mechanics from Aristotle and Ptolemy up through Copernicus, Galileo, and Kepler, and culminating in Newton. Employs simple math and geometry.

[6] Hoffmann, Banesh, *The Strange Story of the Quantum. An Account for the General Reader of the Growth of the Ideas Underlying Our Present Atomic Knowledge*, Dover, 1959. Good non-mathematical survey of this confusing subject that includes the Heisenberg story.

[7] Gardner, Martin, *Relativity Simply Explained*, Dover, 1996. Latest (third) edition of *Relativity for the Million* (1962) and *The Relativity Explosion* (1976). Good non-mathematical explanation of the Special and General Theories of Relativity, which extend Newton's ideas.

[8] Steinmetz, Charles Proteus, *Four Lectures on Relativity and Space* (1923), Dover, 1967, Kessinger Publ, 2005. Excellent presentation of the Special and General theories using elementary mathematics (no tensors, vectors, or calculus).