Kepler's Ellipse

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The Problem: "The Compressed Circle"

I had been exploring how Kepler originally discovered his first two laws and became fascinated by what he did in his *Astronomia Nova* (1609), as presented by a number of researchers. Among the writers was A. E. L. Davis. She mentioned the following regarding Kepler's search for the curve describing the planetary orbits in her *Kepler's Planetary Laws* [Davis 2006]:

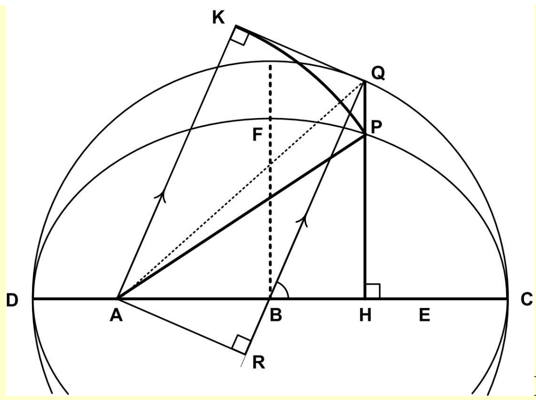

Figure 3

However, such a construction had never been invented before and Kepler did not have the slightest idea what curve the above relationship represented. However, he did know that if the curve were an ellipse, the typical point P would satisfy a vital condition that he had come across in the work of Archimedes: *On Conoids and Spheroids* Prop.4 (where it was stated as if well known even then [1]):

$$PH/QH = BF/BC$$

There is good reason to believe that this was the earliest plane definition of an ellipse, (because it can be derived directly from a section of a cone in three easy steps [Davis 2007]), as well as the definition most commonly, if not exclusively, used by Kepler's contemporaries: it is just the ratio-property of the ordinates. Some people nowadays will recognize it as **the 'compressed circle' property**.

Davis's proof in the July 2007 *Mathematical Gazette* is behind a paywall.[2] The abstract says:

> This article came about as a response to the Supplement to the *Gazette* Number 514 which consists of an extended version of Sir Christopher Zeeman's Presidential Address to The Mathematical Association at York in 2004. … It contains a short section on conics, which seemed relevant to a topic I am already investigating: the level of knowledge of conic geometry potentially available to the Greeks before the time of Apollonius. Professor Zeeman has

[1] JOS: Prop 4: "The area of any ellipse is to that of the auxiliary circle as the minor axis to the major." So the ratio is not what is proved in the Prop. In fact, the ratio is used to *prove* the Prop, that is, it is assumed without proof there, just as Davis suggests (see *On Spheroids* p.113 [Heath 1897, p.303]). So Kepler may have gotten the idea from the proof, but it is not clear where Archimedes got it from.

[2] JOS: After I wrote the original draft of this article and carried out some correspondence with Dr. Davis, Dr. Davis was kind enough to send me a copy of her 2007 paper. I will refer to it later in footnotes, but I decided to preserve the approach I had already taken.

encouraged me to publish the first stage of this research in the *Gazette* to provide a background to his discussion.

Apollonius (of Perga c.262 BC – c.190 BC) is a younger contemporary of Archimedes (c.287 BC – c.212 BC) and both followed Euclid (fl. 300 BC). Heath in his *Works of Archimedes* (1897) [Heath 1897] compares the two as follows (Preface p.*v*):

Michel Chasles has drawn an instructive distinction between the predominant features of the geometry of Archimedes and of the geometry which we find so highly developed in Apollonius. Their works may be regarded, says Chasles, as the origin and basis of two great inquiries which seem to share between them the domain of geometry. Apollonius is concerned with the Geometry of Forms and Situations, while in Archimedes we find the Geometry of Measurements dealing with the quadrature of curvilinear plane figures and with the quadrature and cubature of curved surfaces, investigations which "gave birth to the calculus of the infinite conceived and brought to perfection successively by Kepler, Cavalieri, Fermat, Leibniz, and Newton."

And this (Chapter III. The Relation Of Archimedes To His Predecessors. p.*xxxix*):

… he was no compiler, no writer of textbooks ; and in this respect he differs even from his great successor Apollonius, whose work, like that of Euclid before him, largely consisted of systematising and generalising the methods used, and the results obtained, in the isolated efforts of earlier geometers. There is in Archimedes no mere working-up of existing materials ; his objective is always some new thing, some definite addition to the sum of knowledge, …

Seeking the Solution

In any case, as far as I can discover, it appears that neither Archimedes nor Apollonius (nor Euclid) *proved* the "compressed circle" property of the ellipse. Davis suggested the result follows directly (and easily!) from the conic section definition of an ellipse. Since her approach is hidden, I will have to consider that at some point on my own (see below p.4). Alternatively, I have thought of another way that does not use the conic section definition of ellipse (as far as I know), but it still has gaps.

Desiring an ellipse with semi-major axis a and semi-minor axis b, we take a circle of radius a and rotate it about a diameter as axis by an angle θ (see Figure 1 (a), only semicircles are shown – in blue), where $\cos\theta = b/a$ (see Figure 1 (b)). The resulting figure is sort of like a hard taco shell. Then for any point x along the diameter (measured from the center of the circle) raise a perpendicular

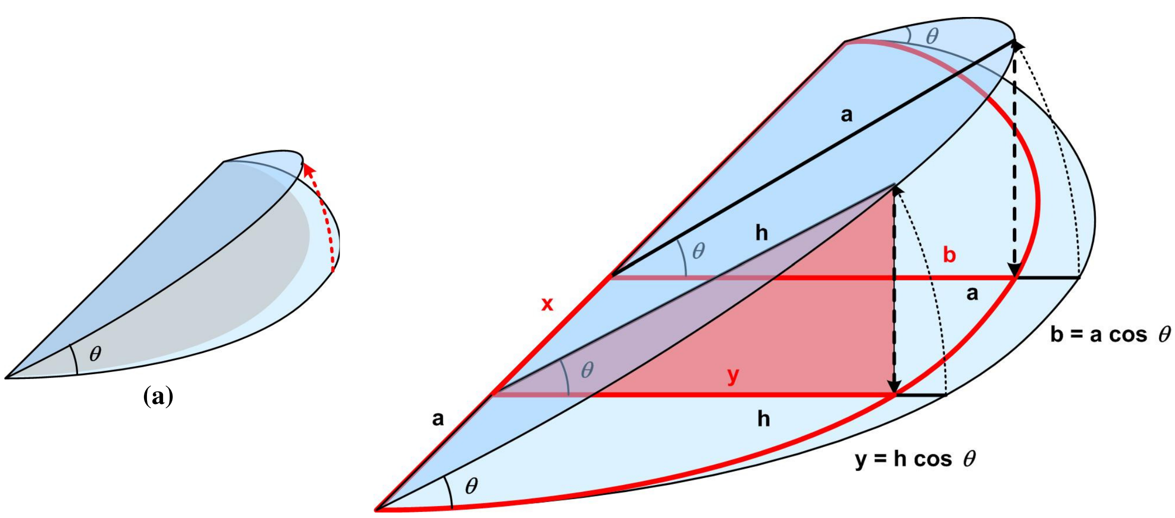

Figure 1 Compressed Circle Curve

until it intersects the circle and designate its length h. Then project the rotated image of h back down onto the original circle. The projected point on the ordinate h will be a distance y from the base of the perpendicular. It is easy to see that $y = h \cos \theta = h\, b/a$. Since x was arbitrary, the resulting curve, the locus of points (x, y), represents the original circle of radius a compressed everywhere by a factor b/a.

The problem at this point is that we do not know if such a curve is in fact an ellipse. I have searched to see if the ancients already noticed that the projection of a rotated circle would yield an ellipse, and I could find no corroboration.

My Construction: Compressed Circle ⇔ Ellipse Parametric Equations

Now it happens that the curve *is* an ellipse, but we arrive at that conclusion from this construction via analytic geometry and a parametric representation of the curve, which is probably anachronistic at this stage, since coordinate geometry (from Descartes 1596–1650) arose towards the end of the period of Kepler's work, almost 2000 years after Archimedes and Apollonius. This can be seen via an augmentation of Figure 1 (see Figure 2).

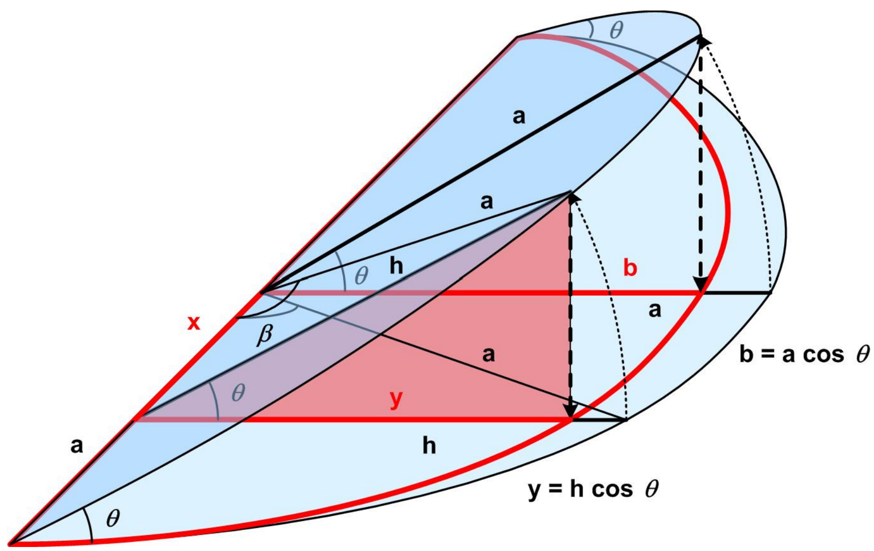

Figure 2 Basis for Parametric Equations for an Ellipse

Draw a radius from the center of the circle to the point where the ordinate h intersects the circle. Let β be the angle this radius makes with the diameter. Then $h = a \sin \beta$, so $y = h\,(b/a) = b \sin \beta$. Similarly $x = a \cos \beta$. So we have the parametric equations:

$$x = a \cos \beta$$
$$y = b \sin \beta$$

It is easy to see that $x^2 / a^2 + y^2 / b^2 = 1$, which is the standard (un-translated, un-rotated) coordinate definition of an ellipse with semi-major axis a and semi-minor axis b.

And conversely, clearly $y / h = b / a$, which proves this parametric formulation of the ellipse satisfies the compressed circle property. So the parametric equation formulation for an ellipse is logically equivalent to the compressed circle property. In fact Davis in her discussion of Kepler argues in this direction, from compressed circle to parametric equations. So to avoid circular reasoning, we must have another way of showing the compressed circle curve is an ellipse, preferably using plane geometry of the Greeks and not analytic geometry of the 17th century. Davis argues from the conic section definition, but somehow I would like to use the rotated circle formulation directly. So that is my gap at the moment.

Rotation Angle θ

One final note about the parametric equations. Figure 3 shows more explicitly the setting for the parametric equations derived from the previous figures. It also shows where the foci of the ellipse, F_1 and F_2, are located and their relationship to the semi-axes of the ellipse. From the relationship $\cos\theta = b/a$ used in the rotated circle generation of the ellipse, we see that the angle θ also appears as shown in Figure 3. (Since any point on the perpendicular bisector of the diameter of the circle is equidistant from both foci and since the sum of the distances from any point on the ellipse to the foci is $2a$, the point on the semi-minor axis of the ellipse is a distance a from each focus.)

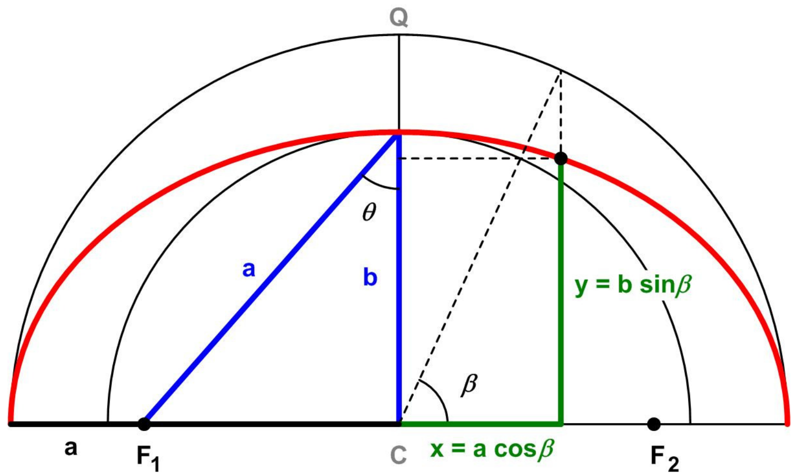

Figure 3 Parametric Equation Diagram

According to an article by J. Hwang [Hwang 1998], when QC was perpendicular to the diameter, Kepler mistakenly thought the angle $\angle CQF_1$ was θ. (In general, as Q moved around the circle, he called $\angle CQF_1$ the *optical angle*.) Given how close the Martian orbit was to a circle (eccentricity $e = 0.093$ (*Wikipedia* 2016)) it is not surprising perhaps that he would make this mistake (approximation). For this value of eccentricity we get the compression factor $\cos\theta = b/a = (1 - e^2)^{½} \approx 0.9957$. According to Hwang, Kepler found $\theta = 5° \ 18' = 5.3°$ and $\cos 5.3° = 0.9957$. So Kepler's result agrees with our current estimate, at least to four decimals.

Solution Found

Ellipse Conic Section ⇒ Compressed Circle (Plane Geometry)

Frustrated in my attempts to prove the compressed circle property of ellipses directly from the conic section definition using only arguments from plane geometry or from my "rotated circle" construction, I returned to Kepler's *Astronomia Nova*, in particular Chapter 59 Protheorem I where Kepler is supposed to use the compressed circle property to prove his construction is an ellipse (see below p.8). My Latin is not good enough, so I finally decided to acquire Donahue's translation [Donahue 2015]. Kepler provided some clues in his statement of the compressed circle property of ellipses (Protheorem I), namely,

> *Using Book I page 21 of the* Conics *of Apollonius, Commandino proves this in his commentary on Proposition 5 of Archimedes'* On Spheroids.

Donahue says the reference cited is actually "Book I *Proposition* 21" in Apollonius's *Conics* (and for some reason Kepler cites Prop 5 in Archimedes instead of Prop 4, which is the correct one, at least according to Heath). But mentioning that Commandinus in his 1566 translation of Apollonius's *Conics* had proved the results was another clue. Alas, I could find no reference for it.

That left only the citation of "Book I Proposition 21" in Apollonius's *Conics*. So I obtained Heath's translation of that work [Heath 1896]. At first I could not make any sense out of it or how it was relevant. For an ellipse, it basically says [Heath 1896, p.19]:

> **Prop 8 [I. 21.]** If AC is the diameter of ellipse AMC, and if M is any point on the circumference of the ellipse, and further if a perpendicular is dropped from M down to the diameter AC intersecting at point L, then
>
> $$\mathbf{ML^2 \propto AL \cdot LC}$$

where "∝" means "is proportional to" and the reference "Prop 8 [I. 21.]" is Heath's numbering with the Apollonius numbering in brackets.

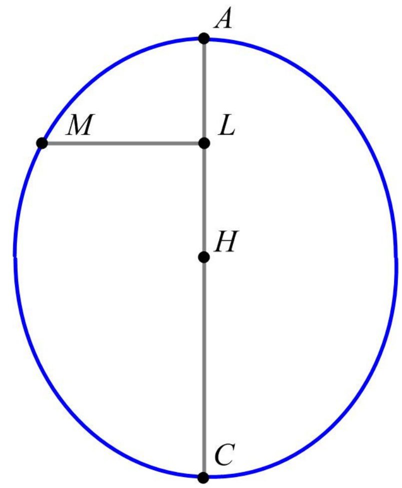

Figure 4 Prop 21 Ellipse

This seemed a strange relationship at first, and certainly not the compressed-circle formulation. The proof of Prop 8 [21] actually relied on figures, constructions, and arguments presented in the preceding propositions. I am still working on the derivation. At one point Apollonius uses a similar but more arcane intermediate relationship to *define the ellipse*. The relationship is clearly derived directly from the conic section form for the ellipse (but not in any "3 easy steps" as suggested by Davis). And so the relationship in Prop 21 above is close to the Greek formulaic definition of the ellipse, but it is not explicitly the compressed-circle formulation.

Then I recalled some plane geometry, and the final compressed-circle ellipse relation popped out effortlessly.

Apollonius Prop 21 Implies Compressed-Circle Ellipse

We add the auxiliary circle AKC to the original ellipse AMC having the same diameter AC. From plane geometry (half angle of inscribed angle on a circle) the angle AKC is a 90° right angle. Then the acute angles α and β are complementary (add up to 90°). But that also means the perpendicular KL divides angle AKC into the same angles α and β. Therefore, triangles AKL and KLC are similar and we have

$$\mathrm{AL : KL = KL : LC}$$

or

$$\mathrm{KL^2 = AL \cdot LC}. \qquad (*)$$[3]

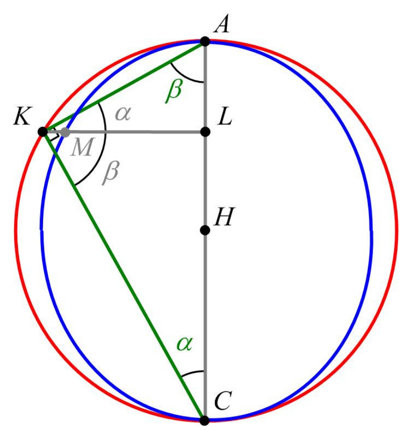

Figure 5 Prop 21 ⇒ Compressed Circle

Now apply the relationship from Prop 21 to get

$$\mathrm{ML^2 \propto AL \cdot LC = KL^2}$$

[3] JOS: It turns out that relation (*) for circles is used extensively by both Apollonius in his proof of Prop 21 and by Davis in her 2007 derivation of the compressed circle relation for an ellipse from its conic section definition. Apollonius cites the relation without proof, but Davis justifies it via Euclid Book III Prop 35. My proof of (*) uses Euclid Book III Prop 20, but it also uses similar triangles, which don't appear until Book VI. That may be a problem, even though Davis also uses similar triangles in her proof later on. So my approach here does not seem that far off.

So ML^2 / KL^2 = constant for any choice of M, in particular when MH is the semi-minor axis b of the ellipse and KH is the semi-major axis a. So ML / KL = b / a for any point M on the ellipse, which is the compressed circle property for the ellipse. At last!

Kepler's Ellipse Construction

Now that the issue of the compressed-circle property of ellipses has been settled, there is another fascinating aspect of the Davis excerpt given at the beginning of this discussion, and it is portrayed in the Figure 3 that accompanies the excerpt. I have reproduced this figure in color with various values attached (see Figure 6). The purpose of the figure was to show how Kepler arrived at the curve for the orbit (of Mars), and how he then computed the distance r from the Sun (focus F_1) to the position of the planet P on the orbit. As Davis indicates, Kepler had been experimenting with various circular arcs centered on the Sun and where their end points fell within the auxiliary circle. With each choice he would compute the corresponding orbit and see how well it fit his data from Tycho Brahe. The choice that fit the data best was the arc shown in Figure 6 intersecting the ordinate to the auxiliary circle at P. This arc was defined by first taking the tangent to the auxiliary circle at Q and then extending a line from the Sun (F_1) parallel to the line with angle β and so perpendicular to the tangent as well. This line was then rotated until the arc generated by its end intersected the ordinate of the auxiliary circle through Q. This point of intersection was the designated point P on the curve representing the orbit.

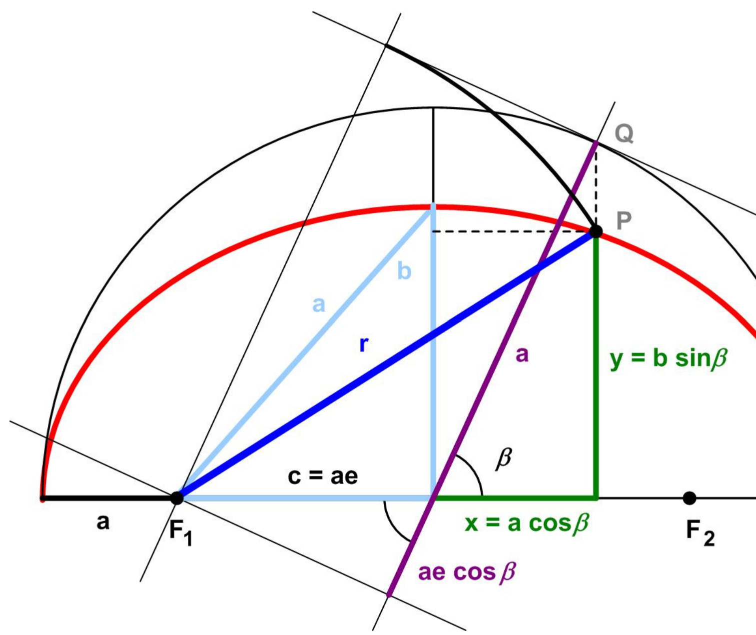

Figure 6 Kepler's Orbit Construction

As Davis said [Davis 2006], Kepler did not know at first what this curve was. "In *Astronomia Nova* Ch. 59, Prop. XI, Kepler set out a rigorous geometrical proof that the typical point he had constructed satisfied the ratio-property which defines an ellipse." In another paper [Davis 1998] Davis provided a modern rendition of this Euclidean plane geometry proof that I will return to later (below p.8).

Kepler Construction ⇒ Ellipse (Analytic Geometry)

I thought I would approach the proof from the point of view of analytic geometry, via the parametric representation for the ellipse and the geometry of the Figure 6. First, what is the equation for r based on Kepler's construction? By construction, r is the radius of a circle which at the point of perpendicularity with the tangent at Q is one side of a rectangle whose opposite side along the line with angle β has an easily computed length (the purple line), namely,

$$r = ae\cos\beta + a = a(1 + e\cos\beta). \tag{1}$$

Now we use the parametric equations for the ellipse to see if we get the same result. From the Pythagorean Theorem we have (using the fact that $b^2 = a^2 - c^2 = a^2 - a^2e^2$)

$$\begin{aligned} r^2 &= (ae + a\cos\beta)^2 + (b\sin\beta)^2 \\ &= a^2(e^2 + 2e\cos\beta + \cos^2\beta) + b^2\sin^2\beta \\ &= a^2(e^2 + 2e\cos\beta + \cos^2\beta) + (a^2 - a^2e^2)(1 - \cos^2\beta) \\ &= a^2(e^2 + 2e\cos\beta + \cos^2\beta + (1 - e^2)(1 - \cos^2\beta)) \\ &= a^2(2e\cos\beta + e^2\cos^2\beta + 1) \\ &= a^2(1 + e\cos\beta)^2 \\ r &= a(1 + e\cos\beta) \end{aligned}$$

which agrees with equation (1)! What a roundabout way to come up with the construction for an ellipse. Why did Kepler ever approach the problem this way?

Epicycle Mindset

Among the material I have been studying is a copy of Kepler's original *Astronomia Nova* in Latin published in 1609 [Kepler 1609]. In addition I have been consulting a website with animations and translated quotes from the original that seeks to describe Kepler's book [LYM 2008]. I haven't digested it all yet, but I came across the motivation for the construction in my Figure 6 above (and Davis's Figure 3). It all came from Kepler's original approach using epicycles and circular motion in general, since that was the prevailing paradigm for almost 2000 years before Kepler.

Following Greek classic theory about heavenly, uniform circular motion and Copernicus's heliocentric theory, Kepler considered the earth and Mars moving in circles around the Sun. But it soon became evident, because of their non-uniform motion, that the planets did not orbit the Sun in circles centered on the Sun. So Kepler considered an (eccentric) circle offset from the Sun. One

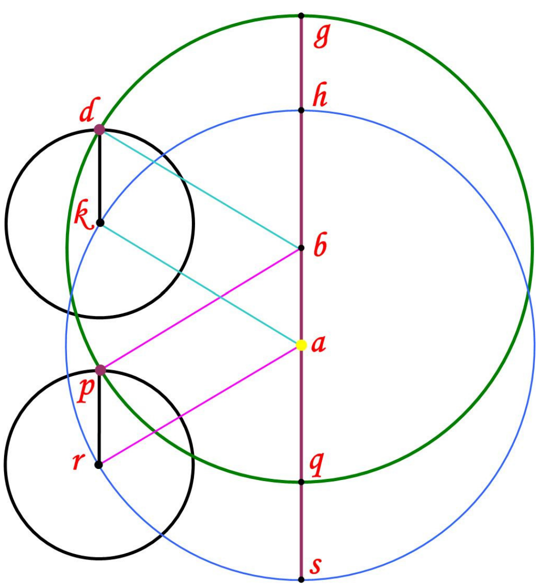

Figure 7 Early Orbit Model with Epicycles

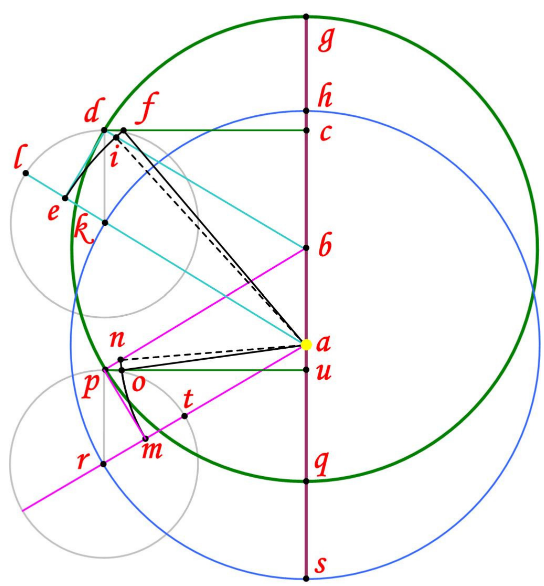

Figure 8 (Colorized) Figure from Chap. 58 *Astronomia Nova*

method of generating it was to have another circle (epicycle) with its center moving around a circle centered on the Sun, with a radius equal to the offset (line between *a* and *b* in Figure 7), and rotating clockwise at the same angular rate as the center rotated counter-clockwise about the Sun. This meant that the radial line from the center of the epicycle to the planet on its rim always maintained the same orientation (vertical in the figure). The resulting orbit for the planet was another circle offset from the Sun circle by the desired amount.

It was with this epicycle in mind that Kepler kept adjusting the position of the planet relative to the epicycle's geometry, in particular via arcs of circles swept from the line drawn from the Sun through the center of the epicycle (Figure 8). Like the Cheshire cat, the epicycle eventually disappeared, but the geometric construct remained to confound me as to its origin. The fact that the intersections of these arcs at points *f* and *n* in Figure 8 generate an ellipse is still remarkable and surprising.

Euclidean Constraints

The epicycles explain some of the construction, but where did the idea of trying to find points by intersecting lines with arcs of circles come from? Why apparently from Euclidean geometry. I have been reading more of Davis's articles and she makes this interesting comment in 1998 [Davis 1998, p.37]:

It was the accepted custom in those days to cite propositions from [Euclid's] *Elements* to validate one's reasoning, but Kepler additionally applied the Euclidean method in a totally different way. Since the principles of Euclidean geometry — the postulates — did not permit numerical measurement of lengths, if a distance were required for a problem its end-points actually had to be constructed in advance, as intersections of straight lines and/or circles. I shall show how Kepler determined the lengths of the radius vectors he required to generate new planetary paths, by identifying their typical points as they turned up in his diagrams, via traditional straight-edge-and-compasses constructions.

Kepler Construction ⇒ Ellipse (Plane Geometry)

After making a number of attempts to prove that Kepler's construction led to an ellipse, using only results from plane geometry instead of analytic geometry, I gave up and resorted to Davis's description of Prop. XI in Chapter 59 of *Astronomia Nova* given in her 1998 paper [Davis 1998, p.42]:

Identification of the curve produced

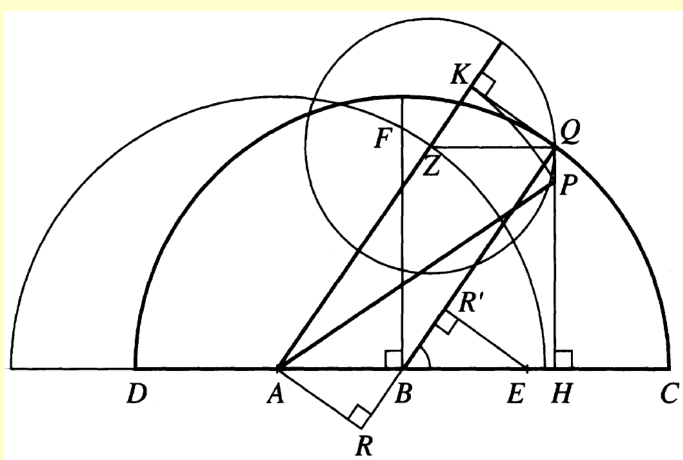

FIGURE 4. Third stage.

Medial-grade orbit: width $BF = b$.

$AZ = BQ = BC = a$, $AB = BE = ae$.
$BQ \parallel AZ$, $AR \parallel KQ$, $\angle QBC = \beta$.
$AP = AK = AZ + ZK$ by construction. Hence also $AP = QR$.

By applying Pythagoras' Theorem (*Elements* I, 47), twice, we obtain:

$$QR^2 = AQ^2 - AR^2$$
$$= (AH^2 + QH^2) - AR^2$$

Now from the similarity of ΔBRA and ΔBHQ (*Elements* VI, 4), we find:

$$\frac{AR}{QH} = \frac{AB}{BQ} = e$$

Hence, $$QR^2 = AH^2 + QH^2 - e^2\, QH^2$$
$$= AH^2 + (1 - e^2)\, QH^2$$

Now from Figure 4, $$AP^2 = AH^2 + PH^2.$$

But $AP = QR$ by construction, which requires that the following condition be satisfied:

$$\frac{PH^2}{QH^2} = 1 - e^2 \quad \text{which is constant.}$$

Now this is precisely the ratio-property of the ordinates by which Kepler had chosen, earlier in Ch. 59 (and in a different context), to define an *ellipse* — so this is the solution to the problem of the path. And the property itself is easily derived in the course of establishing the ellipse as a plane curve, from its original identification as a section of a cone.[4]

… Finally, when we place R' and R symmetrically with respect to B so that $PE = QR'$[5] similarly, it is easy to see that:

$$AP + PE = QR + QR' = 2a,$$

which is the constant distance-sum property that provides the basis for the well-known string-construction for an ellipse: Kepler needed that result in his later work.

That is some impressive geometric argument. I will leave the story for now.

Appendix: Ellipse is Collapsed Circle (Plane Geometry Proof)

(Update 8/2/2018) Apparently the story is not quite finished. I finally came across a purely plane geometry proof that an ellipse is a compressed circle. It turns out not to come directly from Euclid's *Elements*, but rather from an idea put forth in 1822 by the Belgian mathematician Germinal Pierre Dandelin, of Dandelin Sphere fame. Moreover, instead of involving a conic section directly, it employs a section of a cylinder. And instead of a compressed circle, it involves a stretched circle, which turns out to be equivalent.

The insight to the proof came from watching the marvelous Youtube video by Grant Sanderson (3blue1brown [Sanderson 2018]) that demonstrates, using the Dandelin spheres, that the conic section definition of an ellipse is equivalent to the one involving the locus of points in the plane the sum of whose distances from two fixed foci is a constant. I am not going to reproduce that proof in full,

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[4] JOS: As Davis showed explicitly [Davis 2007].

[5] JOS: This takes a bit of manipulation: $\mathbf{QR'}^2 + R'E^2 = QE^2 = QH^2 + HE^2 = QH^2 + (\mathbf{PE}^2 - PH^2) = QH^2 + \mathbf{PE}^2 - (1 - e^2)QH^2 = \mathbf{PE}^2 + e^2 QH^2 = \mathbf{PE}^2 + AR^2 = \mathbf{PE}^2 + R'E^2 \Rightarrow \mathbf{QR' = PE}$. Amazing!

which can be found in the video and in *Wikipedia* (https://en.wikipedia.org/wiki/Dandelin_spheres), but I will discuss it a little bit.

As shown in Figure 9 from *Wikipedia*, we begin by placing a small sphere in the top of the cone and then expanding it until it just touches the ellipse in the intersecting plane at what turns out to be a focus. Similarly we place a large sphere in the bottom of the cone and shrink it until it just touches the intersecting plane at what turns out to be the other focus of the ellipse. These spheres are called the Dandelin spheres.

As Sanderson in the 3blue1brown video showed, the Dandelin sphere idea can be extended to show that the oblique section of a circular cylinder also yields an ellipse. (Just expand the point S in Figure 9 into a circle at the top that matches the circle at the bottom, thus making a cylinder with equal Dandelin spheres still touching the two foci.)

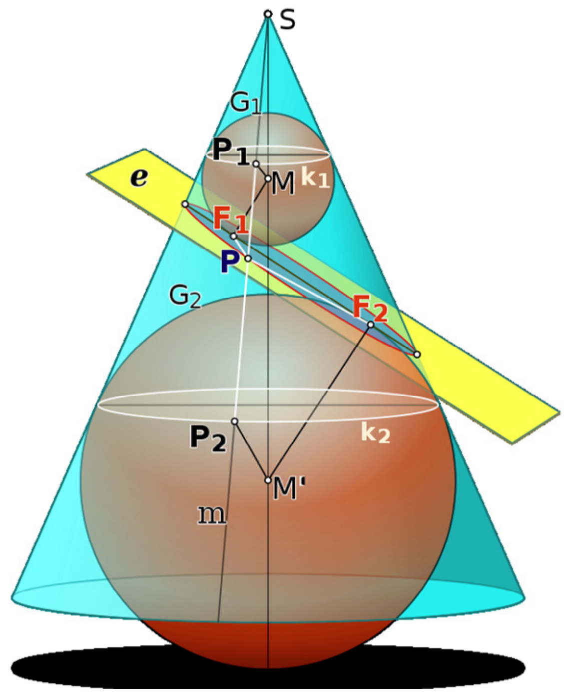

Figure 9 Dandelin Spheres and Ellipse Conic Section (*Wikipedia*)

"Stretched Circle" ⇒ "Compressed Circle"

Sanderson then said the ellipse in the cylinder was a stretched version of a circle. I decided to show this explicitly, as well as how that stretched circle would be equivalent to a compressed circle. Figure 10 shows the ellipse obtained from slicing a plane through the cylinder of (red) radius b (equal to the semi-minor axis of the ellipse). The semi-major axis a is also shown in red in the figure. Now take any point P on the ellipse and drop a perpendicular line to the major axis (blue y), intersecting the axis at a (green) distance x from the center of the ellipse. Slice a vertical plane through the line segment y intersecting the radius b of the base circle at a (green) distance z from the center of the

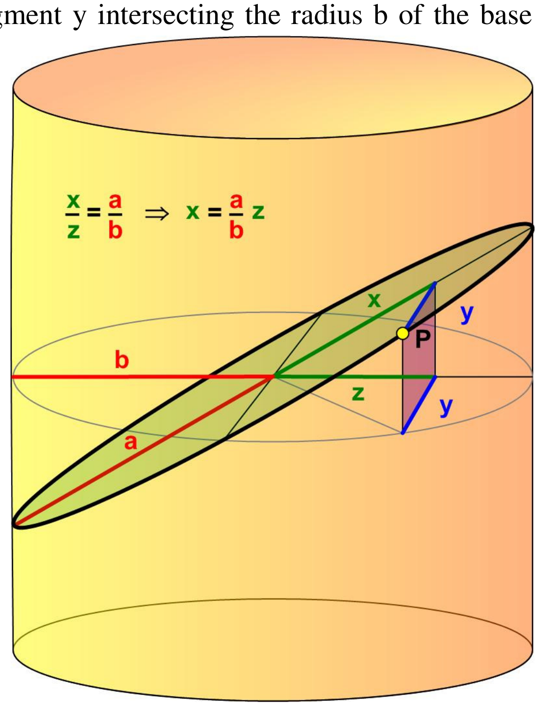

Figure 10 Stretched Circle Ellipse

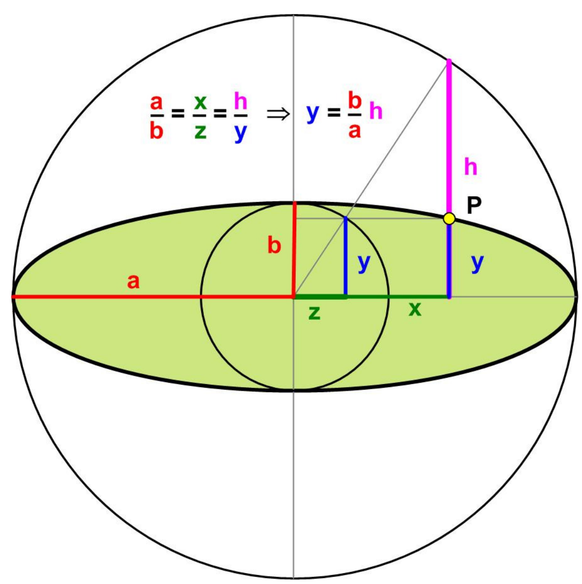

Figure 11 Equivalent Compressed Circle Ellipse

circle and ellipse. The line segment y is projected down onto the circle in a parallel line segment of the same length y. By similar triangles we have a/b = x/z, which implies x = (a/b)z. Since a > b in general, this shows x is a stretched version of z.

Figure 11 shows the typical setup for defining the parametric equations for the ellipse (cf. Figure 3 above). This shows the stretched circle idea more clearly (using the circle of radius semi-minor axis b). If h represents the vertical distance (ordinate) from x to the large circle of radius a, then again by similar triangles we have h/y = x/z. When we combine that with the previous x/z = a/b, we get y = (b/a)x, which is the original compressed circle form for the ellipse (now using the circle of radius semi-major axis a).

Historical Question

Even though Euclid and Apollonius did not show the compressed circle characterization of the ellipse, what about the stretched circle property? If they showed that, then it would be trivial to obtain the compressed circle characterization as I have shown. What this amounts to is showing first that the section of a cylinder is an ellipse.

From Heath this was apparently not well-known for some 600 years [Heath 1921]. He mentions that the mathematician, Serenus of Antinouplis (c. 300 – c. 360 AD),

> observes that many persons who were students of geometry were under the erroneous impression that the oblique section of a cylinder was different from the oblique section of a cone known as an ellipse, whereas it is of course the same curve. [Heath 1921, p.519]

As Heath's discussion is summarized in *Wikipedia*,[6] "Serenus wrote a commentary on the *Conics* of Apollonius, which is now lost. … But he was also a prime mathematician in his own right, having written two works entitled *On the Section of a Cylinder* and *On the Section of a Cone*, works that came to be connected to Apollonius' *Conics*. This connection helped them to survive through the ages." In his work on the cylinder, Serenus used Euclidean methods to show that the sections of a cylinder were indeed ellipses. So by the 4th century AD a simplified Euclidean-type plane geometry proof of the compressed circle property would be available.

Even so, the Dandelin sphere proof is a lot simpler and more elegant than a Euclidean-type argument. My only disappointment is that I still do not have a pure plane geometry proof using my rotated circle (taco shell) argument.

References

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